Non-Linear Buckling Analysis of Columns on Elastic Foundation

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DEDICATION

To All Whom I Love and Appreciate:

To My Parents,

Brothers and

Sisters.

ACKNOWLEDGEMENT

I would like to express my deepest gratitude and appreciation to my supervisor *Dr. Mazen Al-Qaisi* for all the help he offered, the instructions he gave, and guidance during my research work.

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Great appreciation with love is expressed to my dear friend Eng. Rami Al-Tarawnah for his support. I am completely indebted to all my dear friends and my family members, for their encouragement and trust.

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NOMENCLATURE

A_i,C_i	The ith power series coefficients.
C	The mid point deflection of the column.
D_n	The quadratic series of the first order derivative of the power series polynomial coefficients.
dx	The straight element of the unbuckled column, (m).
ds	The curved element of the unbuckled column, (m).
E	The Young's modulus of elasticity of the column material, (N/m^2).
Error %	The percentage error between the buckling loads (p_2) and (p_3).
F_n	The fourth order power series of the power series deflection polynomial coefficients.
h	The length of each sub-interval in the column domain, h=1/n.
I	The moment of inertia of the column cross-section, (m^4).
i,j,k,m,n	Indices.
K_1	The linear modulus of foundation, (N/m^2).
K_2	The quadratic modulus of foundation, (N/m^3).
K_3	The cubic modulus of foundation, (N/m^4).
k_1	The non-dimensional linear modulus of foundation.

k_3 The non-dimensional quadratic modulus of foundation.

k_{eq} The non-dimensional equivalent foundation modulus (linear and non-linear parts).

L The length of the column, (m).

m The mode shape number.

N The approximating function order.

n The number of finite elements along the column domain.

P The buckling load of the column, (N).

p The non-dimensional buckling load of the column.

p_1, p_2, p_3 The non-dimensional buckling load of the columns using different forms.

Q_n The quadratic series of the power series deflection polynomial coefficients.

q(w) The force per unit deflection of the elastic foundation, (N/m).

R The radius of curvature of the column before deformation (m).

$\ddot{R}$ The radius of curvature of the column after deformation (m).

R_n The quadratic series of the second derivative of the power series deflection polynomial coefficients.

Starting C value The value of the first C coefficient in the power series polynomial.

U_B The strain energy due to bending, (N.m).

U_{EF} The strain energy of the elastic foundation, (N.m).

U_P The load potential, (N.m).

U_S The strain energy due to stretching, (N.m).

V The total potential energy functional, (N.m).

V_i The ith energy functional of the problem, (N.m).

V_n The cubic series of the power series deflection polynomial coefficients.

w The transverse deflection of the column.

Greek Symbols

α The ratio factor of non-linear to linear elastic foundation, $\alpha=k_3/k_1$.

ε The axial strain of the column.

χ The change in curvature, (m^{-1}).

θ,ψ The slope of the deflected beam.

κ_1 The non-dimensional linear foundation modulus using the finite difference method, $\kappa_1 = k_1 h^4$.

κ_3 The non-dimensional cubic foundation modulus using the finite difference method, $\kappa_3= k_3 h^4$.

δV	The first variation of the functional.
ζ	The column non-dimensional coordinate.
ζ_i	The starting node coordinate for the ith finite element.
ζ_{i+1}	The end node coordinate for the ith finite element.

ABSTRACT

NON-LINEAR BUCKLING ANALYSIS OF COLUMNS ON ELASTIC FOUNDATION

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In this research the problem of non-linear buckling of columns on elastic foundation was studied using the power series and trial function methods. The clamped-clamped columns were analyzed by both methods, while the trial function method was used to analyze the simply supported columns. A variational principle was employed to derive the differential equation and the boundary conditions of the system. This equation and boundary conditions were reduced to non-dimensional form.

For both solution methods used, a parametric study was conducted. In this study the effect of the linear foundation modulus, ratio factor, the deflection amplitude and the power series polynomial order were studied over the ranges which are expected to be physically valid.

According to the convergence study, the buckling load was determined after a certain power series polynomial order which was noticed to be acceptable at a value of 20.

It was found that the increase in the linear foundation modulus and the decrease in the ratio factor will increase the required buckling load of the structure. It was also concluded that the trend of solution of both power series and trial function method is identical. The power series method was found to be a very powerful method when it was computerized.

Chapter One

LITERATURE SURVEY

1.1 INTRODUCTION.

The elementary theory of the bending of a column on an elastic foundation assumes that the beam is resting on a continuously distributed set of springs the stiffness of which is defined by an equivalent modules of the foundation K (linear and non-linear terms). However, it is rather the exception than the rule that the foundation is actually constituted this way. Generally, the foundation is an elastic continuum characterized by two elastic constants, a modulus of elasticity E and a Poisson's ratio ν.

In certain applications, a column of relatively small bending stiffness is placed on an elastic foundation and loads are axially applied to the column. The loads are transferred through the column to the foundation. The column and foundation must be designed to resist the loads without failing. Often, failure occurs in the column before it occurs in the foundation. Accordingly, in this thesis we assume that the foundation has sufficient strength to prevent its own failure. Furthermore, we assume that the foundation resists the loads transmitted by the column, in a non-linearly elastic manner; that is, the pressure (stress) developed at any point between the column and the foundation is proportional - non-linearly - to the deflection of the column at that point. Since we consider small displacement, the solution presented in this thesis for the column on an elastic foundation is generally conservative for the range of deflection treated.

The nonlinear structural behavior of columns is an important feature in many engineering applications, so problems of elastically supported columns pertain to columns for which the elastic support is provided by a load-bearing medium distributed continuously along the length of the columns. This load-bearing medium is refereed to as foundation. Columns on elastic foundation have a wide application in many technical problems: shells, thin-walled tubes, domes, and beams, to mention just a few. The elastic foundation in such cases is supplied by the resilience of the adjoining portions of a continuous elastic structure. For example, bending analysis of spherical shells can be reduced to the problem of flexure of elastically supported curved columns where the elastic foundation is represented by the resilience of the hoop elements in the shell.

The non-linear structural behavior of columns is consequently the subject of numerous publications. By contrast very few deal with the nonlinear behavior of columns on an elastic foundation. The polynomial approach (power series approach) will be used, since exact and rigorous solutions are extremely hard to obtain. It is well known that the stability problem and the convergence rate of an approximate solution is affected by the nature of the displacement function chosen.

1.2 PAPERS REVIEW.

Many papers available in the literature that deal with beam (and little for columns) on elastic foundation. These beams are subjected to a different loading conditions and different solution procedures have been used to solve them for different boundary conditions.

Elishakoff *et al.*,(1994), they have investigated buckling of initial imperfection sensitive structure - column on a nonlinear elastic foundation. A criterion based on the concept of "modal buckling load" was proposed to determine which modes should be included in the analysis when the weighted residuals method was utilized to calculate the limit load-maximum load the structure can support-for a given initial deflection.

For stochastic analysis, a random field is suggested for the uncertain initial imperfection, and Monte Carlo simulations were performed to obtain the probability density of the buckling load and the reliability of the column.

Finally, a non-stochastic convex model of uncertainly was employed to describe a situation when only limited information was available on uncertain initial deflection, and the minimum buckling load was obtained for this model. The results from both the stochastic and the non-stochastic approaches were derived and critically contrasted.

Amazigo (1971), has obtained an approximate asymptotic expressions for the buckling stresses and auto-correlation of the lateral displacement of infinitely long imperfect columns resting on nonlinear elastic foundations with known autocorrelation. The formulas were discussed and compared with previous results obtained by means of truncated hierarchy and equivalent linearization techniques.

Videc and Sanders (1976), have obtained an approximate asymptotic expression for the buckling load of an imperfect column resting on a nonlinear elastic foundation. The result holds for a large range of imperfection shapes, which were assumed to be stationary random functions of position.

The asymptotic analysis was based on application of Khas'minskii's limit theorem to equations for the slowly varying part of the deflection of the column. Previous results obtained for Gaussian imperfection shapes were shown to be valid also for the larger class of random imperfections considered in this work.

Naschie *et al.*,(1989), used several methods of solution which were chosen on the basis of being coherently connected to the energy method. The method of trial function, an obvious example for the inter-relationship between energy formulation and the solution of the problem, was considered first. Subsequently, the discrete elements method and the method of finite difference were presented. Finally, the Poincare-Lindstedt perturbation and discrete version of this method were considered.

In the work of Sheinman and Adan (1991), the imperfection sensitivity of a beam on a non-linear elastic foundation was studied. The nonlinear equilibrium equations were based on high-order nonlinear kinematic model which takes into account transverse shear deformation. The resulting differential equations were solved by Newton's method and a special finite-difference scheme. A parametric study of the effects of imperfection shape and amplitude was presented for a compression-load isotropic beam on a hardening and softening elastic foundation.

The results indicated that the beam on a hardening foundation exhibits substantial post-buckling stiffness as compared to the beam with a softening foundation. Moreover, it was shown that the overall behavior can be characterized by wavelength changes of the buckled pattern as the loading parameter increases.

Sundararjan (1974), has studied the stability of columns on Winkler type elastic foundations subjected to stationary forces (conservative or non-conservative). Various cases were discussed and a theorem on the influence of the foundation of the critical load was derived.

Smith and Herrmann (1972), have studied an intuitively unexpected and seemingly unknown aspect of the behavior of a cantilevered beam on an elastic foundation subjected to a follower force at its free end. It was found that the critical load for flutter is independent of the foundation modulus which characterizes the Winkler-type elastic embedding. The frequency of vibration of the beam increases with increasing foundation modules, but the magnitude of the critical load is not affected. This result is valid for any value of " tangency coefficient ".

In the work of Hansen and Roorda (1974), the concept of almost sure sample stability and sample stability in probability were formulated for elastic systems. Using a Koiter type approach, these concepts were used in the analysis of imperfection sensitive structures. The applied load and the initial geometric imperfections were introduced into the analysis as random quantities. A compressed beam of finite length on a nonlinear elastic foundation was used in an approximate calculation.

Sheinman *et al.*,(1993), have studied the influence of the choice of approximating function for the axial displacement in nonlinear analysis of isotropic simply supported beams on an elastic foundation.

Three types of functions, namely the vibration, buckling and polynomial approaches, were examined with the aid of the Rayleigh-Ritz procedure, and the polynomial approach function was found to yield the best results, while the buckling approach turned out to be inappropriate for the post-buckling behavior.

Panayotounakos (1989), has investigated a closed-form solution of the nonlinear analysis of a thin and long straight bar, lying on an elastic foundation. This investigation was achieved by means of a strict mathematical procedure concerning the solution of the strongly nonlinear differential equation:

$$\theta'''' + 3\theta'\theta''' \tan\theta + (1 + 3\tan^2\theta)\theta'^2\theta'' + \theta''^2 \tan\theta + (\lambda_1\theta'' / \cos\theta) +$$
$$(\lambda_2\theta'^2 \tan\theta / \cos\theta) = \lambda_3 \sin\theta \cos^2\theta \qquad ;(\lambda_1, \lambda_2, \lambda_3 = \text{constants}),$$

which governs the deflected elastica of the structure. This solution was constructed for values of slope θ lying inside the interval [0°, 40°], because this interval is of practical interest for engineering structures, and, furthermore, the proposed approximations were accurate enough for that interval. Several functional transformations were used and a quantitative analysis was developed yielding reliable results, in conformity with the physical problem.

1.3 Problem Statement

Structures supported by elastic foundation are quite common in engineering, and the literature on the linear analysis of the beam is extensive. Much less coverage has been given to the non-linear analysis of this class of structures, and, in particular, very little attention has been given to structures supported by a non-linear elastic foundation. So, a complete parametric study addressing the effects of a non-linear elastic parameter on the overall non-linear behavior is unavailable. The present work is an attempt to remedy this shortage of information.

It is well known that the foundation status determines whether the structure is in hardening or in softening. So this will address the effect of linear foundation modulus (k_1), ratio factor (α), amplitude (C) and polynomial order (N) on the overall behavior of columns supported by a foundation that exhibits non-linear hardening and softening.

In this work, the problem of non-linear buckling of a column on elastic foundation will be studied and analyzed by using the power series and trial function methods. The governing equation of the structure is a fourth-order non-linear eigenvalue diffferential equation with constant coeffeciant. A clamped-calmped and simply supported columns will be discussed in this work. A power series method is found to be a powerfull method for solving this kind of non-linear porblems. A good agreement in the trend of solution is found to be between the trial function and the power series method.

Chapter Two

FORMULATION OF THE NON-LINEAR BUCKLING PROBLEM OF A PERFECT FINITE COLUMN ON ELASTIC FOUNDATION.

2.1 INTRODUCTION

The first complete and correct solution of stability problem is that of an elastic column by the eminent mathematician Leonard Euler, so the " Euler Elastica" is a well known term in structural engineering. The solution was intended as an illustration of his newly developed calculus of variations. Meantime, we know that this was just the beginning of a new discipline, the theory of post-buckling. This theory has in recent years acquired a profound importance in modern technology of structural engineering design. In this chapter, a variational principle will be used to formulate the non-linear buckling problem of a perfect (exactly straight with no initial deflection) finite column on elastic foundation.

2.2 REMARKS TO THE FORMULATION AND THE SOLUTION OF PROBLEMS IN ENGINEERING.

It is very important, especially at the beginning of a serious study of applied mechanics and science in general, to distinguish clearly between the formulation and the solution of a problem. This is as important as being clear about the physical implication of the assumption underlying a theory. We take it for granted here however that theory, reality and the role of experimental verification are well defined.

Formulating a problem is in fact asking questions whereas the solution is answering those questions. Asking the question in science means formulating the differential equations governing a certain phenomenon such as the buckling of a column. In this thesis, we have suggested an alternative way of posing the question. That is to say, establishing the energy or work functional.

2.3 ESTABLISHING THE CORRESPONDING POTENTIAL ENERGY FUNCTIONAL OF THE PROBLEM.

The first step will thus be establishing the corresponding potential energy functional of the problem. Other very important aspects which should be kept clearly apart are mechanics, geometry and algebra. The simplest way to explain this point is to consider a specific problem which may be the problem at hand, i.e. The buckling of a simply supported column as shown in Figure (2.1).

The axial extensibility (stretching energy) as well as the shear deformation energy are neglected. This neglection is an engineering approximation based on the fact that energy due to axial strain and shear deformation are very small. The only strain energy left are both the strain energy of bending and the strain energy due to the elastic foundation. The expression for that is (Naschie *et al,* 1989) :

$$U_B + U_{EF} = \int_o^L \frac{1}{2} EI\chi^2 dx + \int_0^L [\int_0^w q(w)dw]dx \tag{2.1}$$

where :

U_B : The strain energy due to bending

U_{EF} : The strain energy due to the elastic foundation.

χ : The change in curvature.

E : The young's modules.

I : The moment of inertia of a cross-section of the column.

L : The Length of the column.

w : The transverse deflection of the column.

q(w) : The force per unit deflection of the elastic foundation, and is given as:

$$q(w)=K_1w+K_2w^2-K_3w^3 \tag{2.2}$$

The parameters K_1, K_2 and K_3 are the linear, quadratic and cubic foundation constants, respectively. For simplicity, the quadratic constant K_2 will be assumed zero.

So the strain energy due to the elastic foundation becomes:

$$U_{EF} = \int_0^L \int_0^w [K_1 w - K_3 w^3] dw dx$$

$$= \int_0^L (\frac{1}{2} K_1 w^2 - \frac{1}{4} K_3 w^4) dx \qquad (2.3)$$

Substitute equation (2.3) into equation (2.1):

$$U = U_B + U_{EF}$$

$$= \int_0^L \frac{1}{2} EI\chi^2 dx + \int_0^L (\frac{1}{2} K_1 w^2 - \frac{1}{4} K_3 w^4) dx \qquad (2.4)$$

The load potential, on the other hand, is equal to the load times the end shortening at the sliding support of the column as shown in Figure (2.1).

Thus,

$$U_p = P\Delta \qquad (2.5)$$

Where :

U_p :The load potential.

P :The axial compressive load.

Δ :The end shortening at the sliding support of the column.

The total potential energy functional is therefore:

$$V = \int_0^L \frac{1}{2} EI\chi^2 dx + \int_0^L (\frac{1}{2} K_1 w^2 - \frac{1}{4} K_3 w^4) dx - P\Delta \qquad (2.6)$$

As we see, in this form, (V) is slightly ill posed. This is, for instance, because we have two unknowns χ and Δ which are dependent on each other.

Also, χ is not a very good choice as an independent variable if we wish to solve the problem. In this form, the boundary conditions of the problems are not clearly defined. For these and other reasons, it is more convenient to express χ and Δ in terms of the displacement vector. On the other hand, to find the change of curvature χ and the end shortening Δ in terms of the displacement vector is a problem which causes, relatively speaking, some difficulties. However, this is a problem of differential geometry and not of mechanics.

As we shall see, it turns out that χ and Δ can be expressed in terms of the lateral displacement component w so that the path is now clear to use, for instance, a trial function for w and solve the problem.

In choosing the trial function for w, we are guided by intuitive mechanical considerations such as : what deflection form do we expect when a beam is loaded at the center, and what would change if the boundary conditions would have been different? However, once a trial function has been chosen, we are back in analysis and algebraic manipulations. That is, so because we have to perform integration and differentiation on V. It is this synthesis between physics (mechanics), geometry and algebra (analysis)

...le,
...lso L after
...nbuckled column dx

and a curved element of a buckled column ds are equal (dx=ds). The slope at any point of the buckled column is ψ. In Figure (2.2), a small element of the buckled column is drawn enlarged.

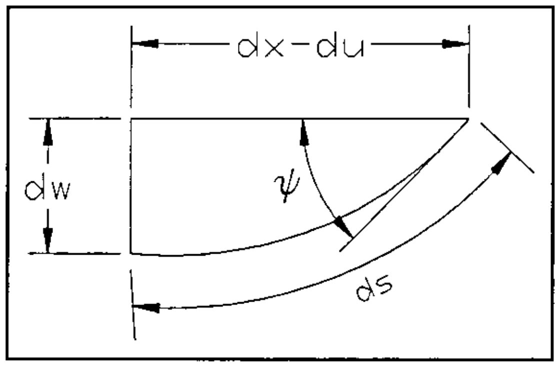


Figure (2.2): Exact elastica of the column

From differential geometry, we know that this element is equivalent to that of Figure (2.3) since the difference between the curved and straight triangle vanishes when taking the differential limit.

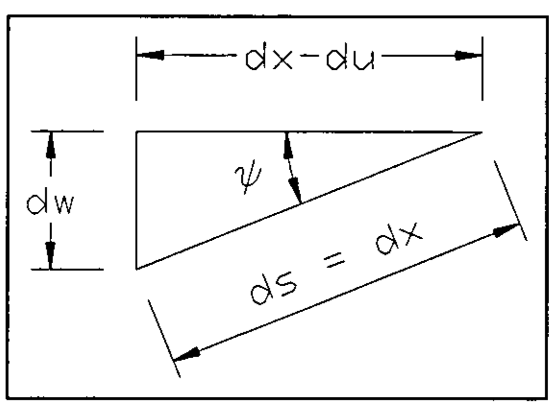


Figure (2.3): Approximate elastica of the column

Note also that the horizontal projection of *ds=dx* is *dx-du*. This is because the horizontal length shrank due to the u displacement component and is no longer L but L-Δ. In order to find the curvature of the buckled column in terms of w, we examine Figure (2.3) from which it is evident that:

$$\sin(\psi) = \frac{dw}{ds} = \frac{dw}{dx} = w'$$

thus

$$\psi = \sin^{-1}(w') \tag{2.7}$$

Since the curvature is nothing but the rate of change of the slope ψ with x, we find that:

$$\frac{1}{\overline{R}} = \psi' = (\sin^{-1}(w'))' = \frac{w''}{(1-w'^2)^{1/2}} \tag{2.8}$$

Noting that the curvature of the column before deformation was :

$$\frac{1}{R} = \frac{1}{\infty} = 0 \tag{2.9a}$$

We see that the change in curvature is:

$$\chi = \frac{1}{\overline{R}} - \frac{1}{R} = \frac{w''}{(1-w'^2)^{1/2}} \tag{2.9b}$$

The attentive reader may now be misled into thinking that this expression is incorrect since he would find the following expression in any book on differential geometry:

$$\chi = \frac{w''}{(1-w'^2)^{3/2}}$$

Also, χ is not a very good choice as an independent variable if we wish to solve the problem. In this form, the boundary conditions of the problems are not clearly defined. For these and other reasons, it is more convenient to express χ and Δ in terms of the displacement vector. On the other hand, to find the change of curvature χ and the end shortening Δ in terms of the displacement vector is a problem which causes, relatively speaking, some difficulties. However, this is a problem of differential geometry and not of mechanics.

As we shall see, it turns out that χ and Δ can be expressed in terms of the lateral displacement component w so that the path is now clear to use, for instance, a trial function for w and solve the problem.

In choosing the trial function for w, we are guided by intuitive mechanical considerations such as : what deflection form do we expect when a beam is loaded at the center, and what would change if the boundary conditions would have been different? However, once a trial function has been chosen, we are back in analysis and algebraic manipulations. That is, so because we have to perform integration and differentiation on V. It is this synthesis between physics (mechanics), geometry and algebra (analysis) which causes difficulties but which also makes structural mechanics so interesting. Mastering this point is probably the key to success in this field.

2.4 THE DIFFERENTIAL GEOMETRY OF THE BUCKLED COLUMN.

Consider the column of Figure (2.1) where the displacement of a point of the unbuckled state p_1 to the buckled state p_2 is indicated. The displacement vector taking p_1,p_2 is thought to be decomposed into two components, the axial displacement component u and the lateral component w which is perpendicular to u.

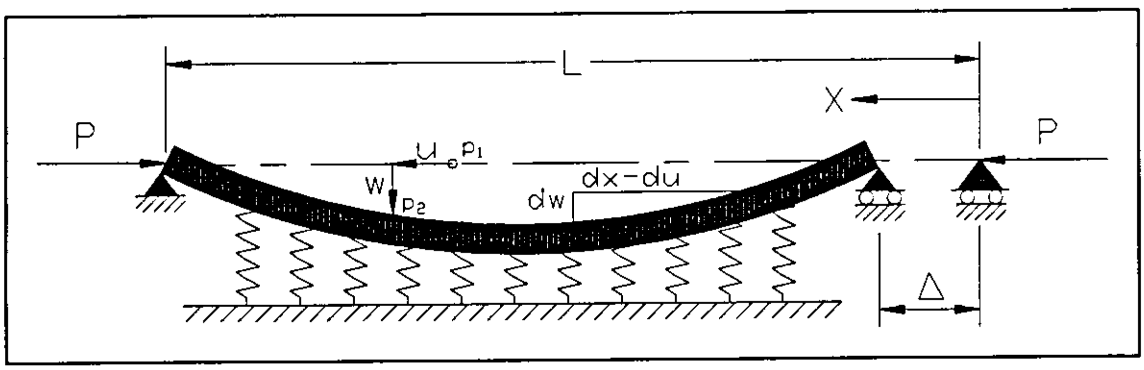
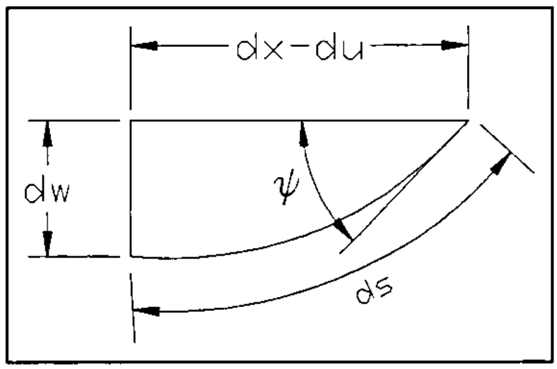


Figure (2.1): Buckling of a finite simply supported column on non-linear elastic foundation.

Here, we are assuring the central line of the column to be inextensible, so that the length L of the column before deformation is also L after deformation. Consequently, a straight element of the unbuckled column dx and a curved element of a buckled column ds are equal (dx=ds). The slope at any point of the buckled column is ψ. In Figure (2.2), a small element of the buckled column is drawn enlarged.



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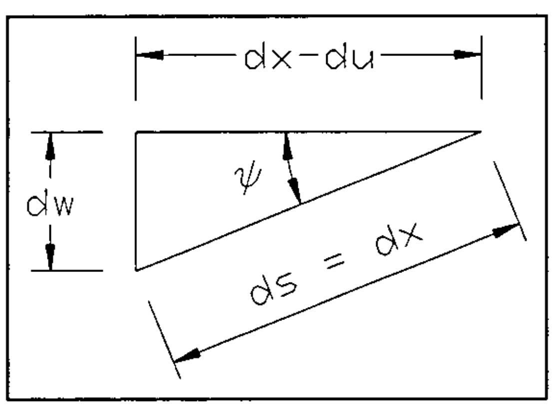

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Note also that the horizontal projection of *ds=dx* is *dx-du*. This is because the horizontal length shrank due to the u displacement component and is no longer L but L-Δ. In order to find the curvature of the buckled column in terms of w, we examine Figure (2.3) from which it is evident that:

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We see that the change in curvature is:

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The attentive reader may now be misled into thinking that this expression is incorrect since he would find the following expression in any book on differential geometry:

$$\chi = \frac{w''}{(1-w'^2)^{3/2}}$$

However, this presumption is not true as the differentiation in both expressions is with respect to different coordinate systems. The last expression does not take into consideration that the column was first straight then moved into buckled state. For the sake of simplicity and consistency we should, therefore, use the expression (2.9b) in finding the strain energy of bending.

The second important geometrical consideration is to express Δ in terms of the displacement vector. To do this, we consider again Figure (2.3). From the theorem of Pythagoras, we see that:

$$(ds)^2=(dx-du)^2+(dw)^2 \tag{2.10a}$$

Dividing by $(dx)^2$ we obtain

$$\left(\frac{ds}{dx}\right)^2=\left(\frac{dx-du}{dx}\right)^2+\left(\frac{dw}{dx}\right)^2 \tag{2.10b}$$

Noting that (ds=dx) and denoting differentiation with respect to x by a prime, we find:

$$1=(1-u')^2+w'^2$$

which means

$$1-u'=\sqrt{(1-w'^2)}$$

or

$$\frac{du}{dx} = 1-(1-w' \ )^{1/2}$$

This leads to

$$du = [1-(1-w'^2)^{1/2}]dx \qquad (2.10c)$$

Integrating both sides and noticing that:

$$\int_0^L du = \Delta \qquad (2.11a)$$

One obtains

$$\Delta = \int_0^L du = \int_0^L [1-(1-w'^2)^{1/2}]dx \qquad (2.11b)$$

The preceding derivations were purely differential geometrical once and did not use any mechanical principles except perhaps the concept of a point moving from p_1 to p_2 which is strictly speaking a kinematical problem.

2.5 THE TOTAL POTENTIAL ENERGY OF THE BUCKLED PROBLEM.

we now come to the physical and mechanical considerations. The total potential energy of the column is defined as the difference between the total strain energy and the load potential:

V = Total Strain Energy - Load Potential

= (Energy due to bending + Energy absorbed from elastic foundation + Energy due to stretching) - Load potential

$= U_B+U_{EF}+U_S-U_P$

$= U-U_P$ (2-12)

Where U can be expressed as :

$$U=\int_0^L \frac{1}{2}EA\varepsilon^2dx+\int_0^L \frac{1}{2}EI\chi^2dx+\int_0^L (\frac{1}{2}K_1w^2-\frac{1}{4}K_3w^4)dx \quad (2.13a)$$

where : ε = the axial strain of the column.

As we have assumed the central line to be inextensible (ds=dx) then the axial strain is zero and so is the stretching energy U_S. Thus, the total potential energy is :

$$V=\int_0^L \frac{1}{2}EI\chi^2dx+\int_0^L (\frac{1}{2}K_1w^2-\frac{1}{4}K_3w^4)dx-P\Delta \quad (2.13b)$$

Inserting the expressions found for χ and Δ in terms of w [from equations (2.9) and (2.11b)] in V, we obtain:

$$V=\int_0^l \left\{\frac{1}{2}EI\frac{w''^2}{(1-w'^2)}-P[1-(1-w'^2)^{1/2}]+\frac{1}{2}K_1w^2-\frac{1}{4}K_3w^4\right\}dx \quad (2.14)$$

Now, after the problem is formulated, we can proceed in two different ways to solve it. We have namely the choice of either of two possibilities, the *first* is to use the calculus of variations to generate the differential equation of the problem from V and then find the solution using numerical integration. The *second* way is to proceed from V directly to the solution.

Before that, we will rearrange V and expand the brackets in order to write the functional in an orderly manner from which, by pure inspection of V, we may make several qualified statements about the expected nature of the solution. Expanding $(1-w'^2)^{-1}$ and $(1-w'^2)^{1/2}$ in power series, we get respectively:

$$(1-w'^2)^{-1} = 1 + w'^2 + w'^4 + \ldots\ldots \tag{2.15a}$$

$$(1-w'^2)^{1/2} = 1 + \frac{1}{2}w'^2 + \frac{1}{8}w'^4 + \ldots\ldots \tag{2.15b}$$

Substitute these expanded forms into equation (2.14) you will get:

$$V = \int_0^L \left\{ \frac{1}{2}EI(w''^2)(1 + w'^2 + w'^4 + \ldots) - P(\frac{1}{2}w'^2 + \frac{1}{8}w'^4 + \ldots) + \frac{1}{2}K_1 w^2 - \frac{1}{4}K_3 w^4 \right\} dx \tag{2.15c}$$

where we have written explicitly terms up to the fourth order.

2.6 Inspection of the total potential energy.

It is usefull for further discussion to write V in the following form:

$$V = V_0 + V_1 + V_2 + V_3 + V_4 + \ldots\ldots \tag{2.16a}$$

where V_i is the sum of all terms of the i^{th} order. For instance, V_0 is of the zero order which means a constant, while V_1 is the sum of all linear terms and V_2 is the sum of all quadratic terms and so on. It should be noted that for the problem at hand :

$V_0=V_1=V_3=V_5=\ldots\ldots.. = 0$ (2.16b)

and we are left with term of even order only, namely:

$$V_2 = \int_0^L (\frac{1}{2}EIw''^2 - \frac{1}{2}Pw'^2 + \frac{1}{2}K_1w^2)dx \quad (2.17)$$

$$V_4 = \int_0^L (\frac{1}{2}EIw'^2w''^2 - \frac{1}{8}Pw'^4 - \frac{1}{4}K_3w^4)dx \quad (2.18)$$

⋮ .

and so on.

It is instructive to understand the meaning of some of the properties reflected in the structure of the V functional. To start with the V_0 constant terms must be and are in fact immaterial to the solution of the problem since variations and differentiation which are required for generating the equilibrium equation can not depend on a constant. A constant simply drops out in a differentiation process.

Second, V_1 must always vanish. This is because stability investigation is, of course, an investigation of equilibrium, which means that the total potential energy must be stationary and we would only like to know whether it is a minimum or a maximum. In other words, the first variation of the functional or the first derivative of the function must vanish.

Third, if we look closely at V_2 we find that it is the well known quadratic form of the classical eigenvalue problem of a buckled column and corresponds to the linearized differential equation:

$$w^{(4)} + \frac{P}{EI}w'' + \frac{K_1}{EI}w = 0 \qquad (2.19)$$

which may be generated from V_2 using the standard method of calculus of variations as we will see later.

As analysis based upon V_2 solely and ignoring all higher order terms (V_4, V_6, ...) would give only the critical eigenvalue buckling load, *but no relationship between the load and the corresponding deflection*. To obtain this relationship, one has to take higher order terms into considerations. This relationship is usually refereed to as the post-buckling curve and is equally obtainable from the corresponding non-linear differential equation:

$$\left[\frac{w''}{(1-w'^2)^{1/2}}\right]'' + \frac{P}{EI}w'' + \frac{K_1}{EI}w - \frac{K_3}{EI}w^3 = 0 \qquad (2.20)$$

Finally, the non existing third order terms (V_3) are an indication that the post-buckling curve has a horizontal tangent at the critical point in a load-deflection plot of the post buckling curve as we will see later.

If we return back to the equations (2.17) and (2.18), we find that the linear foundation constant K_1 is included in the second variation of the functional (linear problem), while the cubic foundation constant K_3 is included in the fourth variation of the functional (non-linear problem), so they are completely separate. If the effect of both constants K_1 and K_3 is to be studied then it is a must to assume a functional of the problem as the sum of both V_2 and V_4 at the same time, i.e.

$$V_{2,4}=V_2+V_4$$

$$\begin{aligned} &= \int_0^L \left(\frac{1}{2}EIw''^2 - \frac{1}{2}Pw'^2 + \frac{1}{2}K_1w^2\right)dx \\ &\quad + \int_0^L \left(\frac{1}{2}EIw'^2w''^2 - \frac{1}{8}Pw'^4 - \frac{1}{4}K_3w^4\right)dx \end{aligned} \tag{2.21}$$

In order that we want to study the effect of both constants K_1 and K_3, then the form of the functional in equation (2.21) should be taken into consideration as it will be shown in chapter four except that the higher order terms of bending energy and load potential and not foundation strain energy will be ignored, i.e. the functional appears in equation (2.21) will be :

$$V_{2,4}^{*} = \int_0^L \left(\frac{1}{2}EIw''^2 - \frac{1}{2}Pw'^2 + \frac{1}{2}K_1w^2 - \frac{1}{4}K_3w^4\right)dx \tag{2.22}$$

2.7 DERIVATION OF THE GOVERNING DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS USING THE CALCULUS OF VARIATION.

The calculus of variation is an important method to transform the form of total potential energy functional to a form of differential equation. The boundary conditions which are required for the solution of this differential equation where assumed to be known. As far as the geometrical boundary conditions are concerned, such as the displacement and the slope of the line of deflection of say a beam, this did not pose any difficulties since these are artificial and predescribed boundary conditions.

However, in the case of the natural boundary conditions, that is to say the dynamical or forced boundary conditions such as the bending moment at the boundary, this could be a somewhat ambiguous situation. This is so because the solution of the differential equations requires a proper number of boundary conditions in order to make the solution unique. For instance, in the case of an elastic column loaded axially, the governing differential equation is of the fourth order and four constants must be found from the boundary conditions. Consequently, four boundary conditions are required. Consider the general case where the potential energy functional is a functional of w'', w' as well as w. We seek to find the first variations of this functional:

$$\delta V = \delta\int_0^L \tilde{V}(w'', w', w, x)dx$$

$$= \int_0^L \delta\tilde{V}dx$$

$$= \int_0^L \left(\frac{\partial\tilde{V}}{\partial w''}\delta w'' + \frac{\partial\tilde{V}}{\partial w'}\delta w' + \frac{\partial\tilde{V}}{\partial w}\delta w\right)dx \qquad (2.23a)$$

where $\tilde{V}$ is as given in equation (2.22) :

$$\tilde{V} = \frac{1}{2}EIw''^2 - \frac{1}{2}Pw'^2 + \frac{1}{2}K_1w^2 - \frac{1}{4}K_3w^4 \qquad (2.23b)$$

The terms with $\delta w''$ and $\delta w'$ must be rewritten using integration by parts into the form including only δw.

Integration by parts gives :

- $\int_0^L \frac{\partial \tilde{V}}{\partial w''}\delta w'' dx = (\frac{\partial \tilde{V}}{\partial w''})\delta w'\Big|_0^L - \int_0^L (\frac{\partial \tilde{V}}{\partial w''})'\delta w' dx$

$$= (\frac{\partial \tilde{V}}{\partial w''})\delta w'\Big|_0^L - (\frac{\partial \tilde{V}}{\partial w''})'\delta w\Big|_0^L + \int_0^L (\frac{\partial \tilde{V}}{\partial w''})''\delta w dx \qquad (2.24a)$$

- $\int_0^L \frac{\partial \tilde{V}}{\partial w'}\delta w' dx = (\frac{\partial \tilde{V}}{\partial w'})\delta w\Big|_0^L - \int_0^L (\frac{\partial \tilde{V}}{\partial w'})'\delta w dx \qquad (2.24b)$

Substitute equations (2.24a,b) into equation (2.23a), you will get :

$$\begin{aligned} \delta V = & \int_0^L [(\frac{\partial \tilde{V}}{\partial w''})'' - (\frac{\partial \tilde{V}}{\partial w'})' + (\frac{\partial \tilde{V}}{\partial w})]\delta w dx \\ & + (\frac{\partial \tilde{V}}{\partial w''})\delta w'\Big|_0^L - (\frac{\partial \tilde{V}}{\partial w''})'\delta w\Big|_0^L + (\frac{\partial \tilde{V}}{\partial w'})\delta w\Big|_0^L \end{aligned} \qquad (2.25)$$

Following the principles of the stationary value of the total potential energy, the equilibrium condition of the column is marked by the vanishing of the first variation of the total potential energy, i.e., by

$$\delta V = 0$$

and since δw is arbitrarily small but non-zero, we get the differential equation:

$$(\frac{\partial\tilde{V}}{\partial w''})'' - (\frac{\partial\tilde{V}}{\partial w'})' + (\frac{\partial\tilde{V}}{\partial w}) = 0 \tag{2.26}$$

We may also write the terms of boundary conditions as :

$$(\frac{\partial\tilde{V}}{\partial w''})\delta w'\Big|_0^L - (\frac{\partial\tilde{V}}{\partial w''})'\delta w\Big|_0^L + (\frac{\partial\tilde{V}}{\partial w'})\delta w\Big|_0^L = 0 \tag{2.27a}$$

where :

$$\left.\begin{aligned}
(\frac{\partial\tilde{V}}{\partial w}) &= K_1 w - K_3 w^3 \\
(\frac{\partial\tilde{V}}{\partial w'}) &= -Pw' \\
(\frac{\partial\tilde{V}}{\partial w'})' &= -Pw'' \\
(\frac{\partial\tilde{V}}{\partial w''}) &= EIw'' \\
(\frac{\partial\tilde{V}}{\partial w''})' &= EIw''' \\
(\frac{\partial\tilde{V}}{\partial w''})'' &= EIw^{(4)}
\end{aligned}\right\} \tag{2.27b}$$

For the sake of clarity, we may write the terms of boundry conditions of equation (2.27a) explicitly as :

$$\begin{aligned}
&[EIw''(L)\delta w'(L) - EIw''(0)\delta w'(0)] - [EIw'''(L)\delta w(L) - EIw'''(0)\delta w(0)] \\
&-[Pw'(L)\delta w(L) - Pw'(0)\delta w(0)] = 0
\end{aligned} \tag{2.27c}$$

Similarly, subsitute equation (2.27b) into equation (2.26) to find the general form of the governing fourth-order non-linear eigenvalue differential equation of a column on elastic foundation:

$$EIw^{(4)} + Pw'' + K_1 w - K_3 w^3 = 0 \tag{2.28}$$

We may look now at the differnt ways of supporting the column and then see how the missing boundary conditions can be found from the condition that the above boundary expression [equation (2.27c)] must vanish for $\delta V = 0$, i.e. for equilibrium.

2.7.1 Boundary conditions of a column with both ends clamped.

For a both ends clamped column the situation is straigh forward because the imposed clamped boundary conditions are clearly that displacement and slope at both ends must vanish which gives us the required four boundary conditions. The boundary conditions of a column shown in Figure (2.4) are clearly :

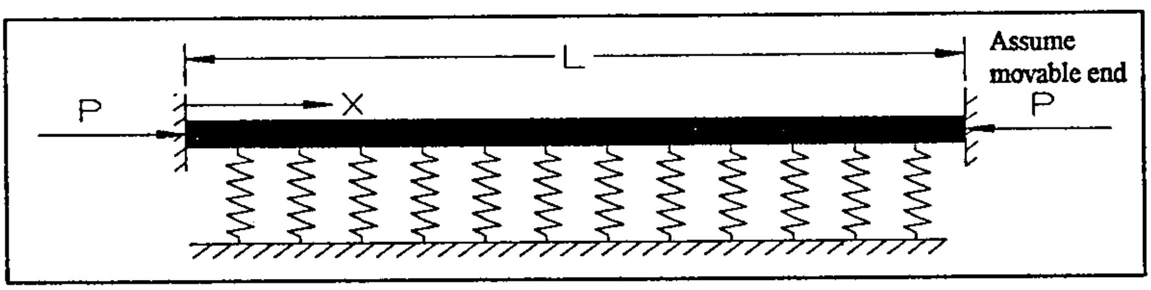


Figure (2.4) : A finite clamped-clamped column on non-linear elastic foundation.

$$\left.\begin{array}{lll} w(0)=0 & , & w(L)=0 \\ w'(0)=0 & & w'(L)=0 \end{array}\right\} \tag{2.29}$$

Since the variation of a constant is zero, it follows from these boundary conditions that:

$$\delta w(0)=0 \qquad \delta w(L)=0$$
$$\delta w'(0)=0 \quad , \quad \delta w'(L)=0$$

and thus all the boundary expressions can vanish. The clamped-clamped boundary condition leads, therefore, to a unique solution and equilibrium.

2.7.2 Boundary conditions of a column with both ends simply supported.

Suppose now that instead of being clamped, the column is merely simply supported at both ends. In this case, the only physically clear boundary conditions are the two prescribed boundary conditions, namely the vanishing of the displacement at both ends of the column and one must now ask from where the two remaining conditions should come. It is a particularly fortunate situation that the calculus of variation always furnishes automatically the correct number of boundary conditions.

The only two boundary conditions of a column shown in Figure (2.5) are clearly :

$w(0)=0$ and $w(L)=0$ (2.30a)

This implies that :

$$\delta w(0)=0 \text{ and } \delta w(L)=0$$

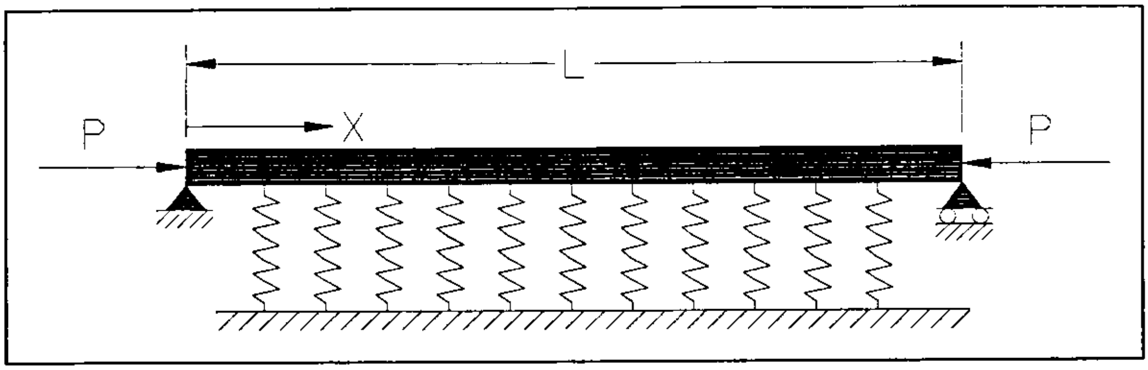


Figure (2.5) : A finite simply supported column on non-linear elastic foundation.

Consequently, and as can be seen clearly from the previous equation for the boundary terms [Equation (2.27c)], we have to make the other boundary conditions vanish by requiring that :

$$w''(0) = 0 \text{ and } w''(L) = 0 \tag{2.30b}$$

and consequently, all boundary expressions can be made zero as should be. In other words, simply supported means that, the bending moment is zero at both ends because $w'' = 0$ means M=0 for a column.

2.8 REDUCTION OF THE GOVERNING DIFFERENTIAL EQUATION, BOUNDARY CONDITIONS AND THE TOTAL POTENTIAL ENERGY FUNCTIONAL TO NON-DIMENSIONAL FORM.

The next step is to write the differential equation, boundary conditions and the functional in a non-dimensional form. If the non-dimensional position ζ is defined such that :

$$\zeta = \frac{x}{L} \tag{2.31a}$$

and from this definition, it follows that:

$$\frac{d\zeta}{dx} = \frac{1}{L} \tag{2.31b}$$

Using the chain rule for differentiation, with the help of equation (2.31b) to find that :

$$\frac{dw}{dx} = \frac{dw}{d\zeta}\frac{d\zeta}{dx} = \frac{1}{L}\frac{dw}{d\zeta} \tag{2.32a}$$

$$\frac{d^2w}{dx^2} = \frac{d}{d\zeta}\left(\frac{dw}{dx}\right)\frac{d\zeta}{dx} = \frac{1}{L^2}\frac{d^2w}{d\zeta^2} \tag{2.32b}$$

$$\frac{d^3w}{dx^3} = \frac{d}{d\zeta}\left(\frac{d^2w}{dx^2}\right)\frac{d\zeta}{dx} = \frac{1}{L^3}\frac{d^3w}{d\zeta^3} \tag{2.32c}$$

$$\frac{d^4w}{dx^4} = \frac{d}{d\zeta}\left(\frac{d^3w}{dx^3}\right)\frac{d\zeta}{dx} = \frac{1}{L^4}\frac{d^4w}{d\zeta^4} \tag{2.32d}$$

2.8.1 Reduction of the differential equation.

Substitute equations (2.32) into the differential equation [Equation (2.28)], one can get:

$$EI\frac{1}{L^4}\frac{d^4w}{d\zeta^4}+P\frac{1}{L^2}\frac{d^2w}{d\zeta^2}+K_1w-K_3w^3=0$$

Divide all terms of the previous differential equation by the non-zero term $\frac{EI}{L^4}$, you will get:

$$\frac{d^4w}{d\zeta^4}+\frac{PL^2}{EI}\frac{d^2w}{d\zeta^2}+\frac{K_1L^4}{EI}w-\frac{K_3L^4}{EI}w^3=0$$

Assume the following non-dimensionalization:

$$\left.\begin{aligned} p&=\frac{PL^2}{EI}\\ k_1&=\frac{K_1L^4}{EI}\\ k_3&=\frac{K_3L^4}{EI}\end{aligned}\right\}\qquad (2.33)$$

The differential equation of its non-dimensional form takes its final shape as:

$$\frac{d^4w}{d\zeta^4}+p\frac{d^2w}{d\zeta^2}+k_1w-k_3w^3=0 \qquad (2.34)$$

2.8.2 Reduction of the boundary conditions.

(1) Clamped- Clamped Ends:

Use equation (2.31a) to find that :

For $x = 0 \quad \rightarrow \quad \zeta = \frac{0}{L} = 0$

For $x = L \quad \rightarrow \quad \zeta = \frac{L}{L} = 1$

So the boundary conditions for the clamped case [Equation (2.29)] become in their non-dimensional form as:

$$\left.\begin{matrix} w(0) = 0 & , & w(1) = 0 \\ w'(0) = 0 & & w'(1) = 0 \end{matrix}\right\} \tag{2.35}$$

(2) Simply supported ends:

Similarly, the boundary conditions for the simply supported case [Equation (2.30a,b)] take their non-dimensional form as:

$$\left.\begin{matrix} w(0) = 0 & , & w(1) = 0 \\ w''(0) = 0 & & w''(1) = 0 \end{matrix}\right\} \tag{2.36}$$

2.8.3 Reduction of the functional.

The total potential energy functional which will be reduced to the non-dimensional from appears in equation (2.22). The non-dimensional form simply looks like:

$$V_{2,4}^{**} = \int_0^1 (\frac{1}{2}w''^2 - \frac{1}{2}pw'^2 + \frac{1}{2}k_1w^2 - \frac{1}{4}k_3w^4)d\zeta \qquad (2.37)$$

In what follows, the problem will only be discussed in non-dimensional form. The qualification phrase " non-dimensional " will be omitted for convenience; for example p is called the external axial load, etc...

Chapter three

SOLUTION OF THE BUCKLED PROBLEM USING THE METHOD OF TRIAL FUNCTION.

3.1 INTRODUCTION

One of the most important applications of the energy methods is in establishing approximate solutions using trial functions. This trial function is generally an approximation to the quantity in which energy functional is expressed. There are, of course , certain conditions which this function must satisfy. To start with, the following should be achieved:

(1) The approximation is as good as our guess of the expected shape of deformation, and

(2) Certain boundary conditions must be satisfied.

In this method, the geometrical boundary conditions, i.e. the boundary conditions associated with certain geometrical quantities, namely deflection (w) and slope (w') must be satisfied. Other boundary conditions, the so called dynamical or force boundary conditions associated with the bending moment and thus indirectly with (w'') are optional and when they are achieved the solution will be more accurate.

3.2 THE APPROPRIATE SELECTION OF THE TRIAL FUNCTION.

3.2.1 Simply supported case:

The suitable trial function related to the simply supported column which satisfies both geometrical and dynamical boundary conditions of equation (2.36) is a sinusoidal wave of the form:

$$w = C\sin(m\pi\zeta) \tag{3.1}$$

where:

w: The transverse deflection of the column on the domain $\zeta \in [0,1]$.

C: The maximum transverse deflection at the mid point of the column where $\zeta = \frac{1}{2}$.

m: The mode shape number [m = integer = 1,2,3,..... etc].

ζ: The non-dimensional position along the column, $\zeta \in [0,1]$.

3.2.2 Clamped-clamped case:

The suitable well known trial function related to the clamped-clamped column and achieve the geometrical boundary conditions of equation (2.35) is a cosine wave of the form:

$$w = \frac{C}{2}[1 - \cos(2m\pi\zeta)] \tag{3.2}$$

where:

w: The transverse deflection of the column on the domain $\zeta \in [0,1]$.

C: The maximum transverse deflection at the mid point of the column where $\zeta = \frac{1}{2}$.

m: The mode shape number [m = integer = 1,2,3,..... etc].

ζ: The non-dimensional position along the column, $\zeta \in [0,1]$.

3.3 THE METHOD OF TIMOSHENKO FOR DETERMINING THE CRITICAL VALUE OF THE COMPRESSIVE LOAD.

The straight form of equilibrium of the compressed column is stable if the compressive load (p) is small, but unstable after (p) reaches its critical value at which lateral buckling begins. This critical value of (p) may be found by comparing the energy of the system in the two cases: (1) When the column is straight and (2) When it is compressed and bent.

The strain energy in the bent column is larger than that in the straight compressed form, because the energy of bending must be added to the energy of compression, which may be considered constant for small lateral deflections. The potential energy of the load (p) must also be considered. Let U be the strain energy due to bending and to elastic foundation and U_P the decrease in the potential energy of the load. Then if U_P is less than U, deflection of the column is accompanied by an increase in the potential energy of the system. This means that it would be necessary to apply some additional lateral force to produce bending. In such a case the straight form of equilibrium is *stable.*

On the other hand, if $U_P > U$, deflection the column is accompanied by a decrease in the potential energy of the system, and the bending will proceed without application of any lateral force, i.e., the straight form of equilibrium is *unstable.* The critical value of the compressive load is therefore obtained from the condition:

$$U = U_P$$

or in other words when functional vanishes, i.e. use equation (2.37) to find that :

$$p = \frac{\int_0^1 \left(\frac{1}{2} w''^2 + \frac{1}{2} k_1 w^2 - \frac{1}{4} k_3 w^4\right) d\zeta}{\int_0^1 \frac{1}{2} w'^2 d\zeta} \tag{3.3}$$

In what follows, different non-dimensional functional forms which were discussed in chapter two, will be used to find the compressive buckling load, these functionals are:

(1) Functional of equation (2.17):

This functional is independent on k_3, i.e. it gives the well known buckling load (p_1) of the classical eigenvalue problem (*linear case*).

(2) Functional of equation (2.21):

This functional is dependent on both k_1 and k_3 and gives the general critical buckling load (p_2) for the (*non-linear problem*).

(3) Functional of equation (2.37):

This functional is the same as that of equation (2.21) but the effect of the high-order non-linear terms of bending energy and load potential are neglected, this will give the critical buckling load (p_3).

In this chapter, it will be shown that it is a good approximation to use the functional of equation (2.37), i.e., [equation (3.3)] rather than equation (2.21) since the percentage error is too small.

3.4 DERIVATION OF THE BUCKLING LOAD FORMS FROM THE PREVIOUS THREE-FUNCTIONAL FORMS FOR BOTH SIMPLY-SUPPORTED AND CLAMPED-CLAMPED COLUMNS ON ELASTIC FOUNDATION.

3.4.1 Functional of equation (2.17) [linear problem].

Apply the method of Timoshenko to equation (2.17), you will get:

$$p_1 = \frac{\int_0^1 (\frac{1}{2} w''^2 + \frac{1}{2} k_1 w^2) d\zeta}{\int_0^1 \frac{1}{2} w'^2 d\zeta} \tag{3.4}$$

(A) Simply supported case:

Applying the deflection shape w in equation (3.1) into equation (3.4), one can find that:

- $\int_0^1 \frac{1}{2} w''^2 d\zeta = \frac{C^2}{2} \int_0^1 [-(m\pi)^2 \sin(m\pi\zeta)]^2 d\zeta = \frac{1}{4}(m\pi)^4 C^2$
- $\int_0^1 \frac{1}{2} k_1 w^2 d\zeta = \frac{k_1 C^2}{2} \int_0^1 \sin^2(m\pi\zeta) d\zeta = \frac{1}{4} k_1 C^2$
- $\int_0^1 \frac{1}{2} w'^2 d\zeta = \frac{C^2}{2} \int_0^1 [(m\pi)\sin(m\pi\zeta)]^2 d\zeta = \frac{1}{4}(m\pi)^2 C^2$

Substitute into equation (3.4), the result will be:

$$p_1 = \frac{(m\pi)^4 + k_1}{(m\pi)^2} \tag{3.5a}$$

(B) Clamped-clamped case:

Apply the deflection shape w in equation (3.2) into equation (3.4), you will find that:

- $\int_0^1 \frac{1}{2} w''^2 d\zeta = \frac{C^2}{2} \int_0^1 [\frac{1}{2}(2m\pi)^2 \cos(2m\pi\zeta)]^2 d\zeta = (m\pi)^4 C^2$
- $\int_0^1 \frac{1}{2} k_1 w^2 d\zeta = \frac{k_1 C^2}{2} \int_0^1 [\frac{1}{2}(1 - \cos(2m\pi\zeta)]^2 d\zeta = \frac{3}{16} k_1 C^2$
- $\int_0^1 \frac{1}{2} w'^2 d\zeta = \frac{C^2}{2} \int_0^1 [(m\pi)\sin(2m\pi\zeta)]^2 d\zeta = \frac{1}{4}(m\pi)^2 C^2$

substitute into equation (3.4), the results will be :

$$p_1 = \frac{16(m\pi)^4 + 3k_1}{4(m\pi)^2} \tag{3.5b}$$

3.4.2 Functional of equation (2.21) [Non-linear problem]

Apply the method of Timoshenko to equation (2.21) in its non-dimensional form you can get :

$$p_2 = \frac{\int_0^1 (\frac{1}{2}w''^2 + \frac{1}{2}k_1 w^2 + \frac{1}{2}w' \ w''^2 - \frac{1}{4}k_3 w^4)d\zeta}{\int_0^1 (\frac{1}{2}w'^2 + \frac{1}{8}w'^4)d\zeta} \tag{3.6}$$

(A) Simply supported case :

Use equation (3.1) to substitute directly for w into equation (3.8) to find that :

- $\int_0^1 \frac{1}{2}w'^2 w''^2 d\zeta = \frac{C^4}{2}\int_0^1 [-(m\pi)^2 \sin(m\pi\zeta)]^2 [(m\pi)\cos(m\pi\zeta)]^2 d\zeta = \frac{1}{16}(m\pi)^6 C^4$
- $\int_0^1 \frac{1}{4}k_3 w^4 d\zeta = \frac{k_3 C^4}{4}\int_0^1 \sin^4(m\pi\zeta)d\zeta = \frac{3}{32}k_3 C^4$ *
- $\int_0^1 \frac{1}{8}w'^4 d\zeta = \frac{C^4}{8}\int_0^1 [(m\pi)\cos(m\pi\zeta)]^4 d\zeta = \frac{3}{64}(m\pi)^4 C^4$ **

---

* From calculus: $\int \sin^n(x)dx = \frac{-1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$

** From calculus : $\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$

Substitute into equation (3.6), the result will be:

$$p_2 = \frac{16(m\pi)^4 + 16k_1 + [4(m\pi)^6 - 6k_3]C^2}{16(m\pi)^2 + 3(m\pi)^4 C^2} \tag{3.7a}$$

(B) Clamped-clamped case:

Similarly, use equation (3.2) to substitute directly for w into equation (3.6) to find that :

- $\int_0^1 \frac{1}{2} w'^2 w''^2 d\zeta = \frac{C^4}{2}\int_0^1 [(m\pi)\sin(2m\pi\zeta)]^2 [\frac{1}{2}(2m\pi)^2 \cos(2m\pi\zeta)]^2 d\zeta = \frac{1}{4}(m\pi)^6 C^4$

- $\int_0^1 \frac{1}{4} k_3 w^4 d\zeta = \frac{k_3 C^4}{4}\int_0^1 [\frac{1}{2}(1-\cos(2m\pi\zeta)]^4 d\zeta = \frac{35}{512} k_3 C^4$

- $\int_0^1 \frac{1}{8} w'^4 d\zeta = \frac{C^4}{8}\int_0^1 [(m\pi)\sin(2m\pi\zeta)]^4 d\zeta = \frac{3}{64}(m\pi)^4 C^4$

Substitute into equation (3.6), the result will be :

$$p_2 = \frac{512(m\pi)^4 + 96k_1 + [128(m\pi)^6 - 35k_3]C^2}{128(m\pi)^2 + 32(m\pi)^4 C^2} \tag{3.7b}$$

3.4.3 Functional of equation (2.37) [Non-linear problem]

The application of the method of Timoshenko was already given in equation (3.3) as :

$$p_3 = \frac{\int_0^1 (\frac{1}{2} w''^2 + \frac{1}{2} k_1 w^2 - \frac{1}{4} k_3 w^4) d\zeta}{\int_0^1 \frac{1}{2} w'^2 d\zeta} \tag{3.8}$$

(A) Simply supported case :

Substitute directly for w to get :

$$p_3 = \frac{16(m\pi)^4 + 16k_1 - 6k_3C^2}{16(m\pi)^2} \tag{3.9a}$$

(B) Clamped-clamped case :

Similarly, substitute directly for w to get :

$$p_3 = \frac{512(m\pi)^4 + 96k_1 - 35k_3C^2}{128(m\pi)^2} \tag{3.9b}$$

For the sake of clarity, we may summarize the previous expressions of the compressive buckling load for different functional forms in Table (3.1) .

Functional equation no.	Ends supporting state	
	Simply supported column	Clamped-clamped column
Equation (2.17)	$p_1 = \frac{(m\pi)^4 + k_1}{(m\pi)^2}$	$p_1 = \frac{16(m\pi)^4 + 3k_1}{4(m\pi)^2}$
Equation (2.21)	$p_2 = \frac{16(m\pi)^4 + 16k_1 + [4(m\pi)^6 - 6k_3]C^2}{16(m\pi)^2 + 3(m\pi)^4 C^2}$	$p_2 = \frac{512(m\pi)^4 + 96k_1 + [128(m\pi)^6 - 35k_3]C^2}{128(m\pi)^2 + 32(m\pi)^4 C^2}$
Equation (2.37)	$p_3 = \frac{16(m\pi)^4 + 16k_1 - 6k_3C^2}{16(m\pi)^2}$	$p_3 = \frac{512(m\pi)^4 + 96k_1 - 35k_3C^2}{128(m\pi)^2}$

Table (3.1) : Buckling-Load expressions for different functional forms for both simply supported and clamped-clamped columns on elastic foundation.

In the all previous expressions of the buckling load, we will only study the case of the first mode, i.e. [m=1]. It should be pointed here that the symbol "α" will be used to express the ratio factor of the cubic constant of foundation to the linear constant of it, i.e. $\alpha = k_3/k_1$. This constant α may be positive (Softening case), or on the other hand, it may be negative (Hardening case).

Chapter Four

SOLUTION OF THE BUCKLED PROBLEM USING THE POWER SERIES METHOD

4.1 INTRODUCTION

In general, the power series method is the standard basic method for solving linear deferential equation with variable coefficients. For the domain $0 \le \zeta \le 1$, the expected form of the series solution is :

$$w(\zeta) = C_0 + C_1\zeta + C_2\zeta^2 + C_3\zeta^3 + \ldots$$

$$= \sum_{n=0}^{\infty} C_n \zeta^n \qquad (4.1)$$

In our problem we will try to solve another kind of differential equations that is, non-linear eigenvalue deferential equation and see how much the validity of applying the power series method for such a problem is. In the power series method we differentiate, add, multiply power series. These three operations are permissible, in the sense explained in what follows.

4.2 SOLUTION BY THE POWER SERIES METHOD (THE RECURRENCE FORMULA)

From sub-section (2.8.1), the equation of buckled system in non-dimensional form for the clamped-clamped column is:

$$\frac{d^4 w}{d\zeta^4} + p\frac{d^2 w}{d\zeta^2} + k_1 w - k_3 w^3 = 0$$

with the boundary conditions:

1. <u>at $(\zeta = 0)$</u> :

$w(0) = 0$, which means that:

$C_0+C_1\times 0 +C_2\times 0+....+C_N\times 0=0 \Rightarrow C_0=0$

also,

$\left.\frac{dw}{d\zeta}\right|_{\zeta=0} = 0$, which means that :

$1\times C_1+2\times C_2\times 0+3\times C_3\times 0+.....+N\times(N-1)C_N\times 0=0 \Rightarrow C_1=0$

2. <u>at $(\zeta = 1)$:</u>

$w(1) = 0$

also,

$$\left.\frac{dw}{d\zeta}\right|_{\zeta=1} = 0$$

For the case of clamped-clamped column on elastic foundation, a power series function which already achives these boundary conditions will be used as it will be shown later.

From equation (4.1) the following can be written:

$$\frac{dw}{d\zeta}=\sum_{n=1}^{\infty} nC_n\zeta^{n-1} \tag{4.2a}$$

$$\frac{d^2w}{d\zeta^2}=\sum_{n=2}^{\infty} n(n-1)C_n\zeta^{n-2} \tag{4.2b}$$

$$\frac{d^3w}{d\zeta^3}=\sum_{n=3}^{\infty} n(n-1)(n-2)C_n\zeta^{n-3} \tag{4.2c}$$

$$\frac{d^4w}{d\zeta^4}=\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)C_n\zeta^{n-4} \tag{4.2d}$$

If the series is truncated by (N) terms, we will have after shifting the index (n) :

$$\frac{d^2w}{d\zeta^2}=\sum_{n=0}^{N-2} (n+2)(n+1)C_{n+2}\zeta^{n} \tag{4.3a}$$

$$\frac{d^4w}{d\zeta^4}=\sum_{n=0}^{N-4} (n+4)(n+3)(n+2)(n+1)C_{n+4}\zeta^{n} \tag{4.3b}$$

The term w^3 can be expressed in series form as:

$$w^3=\left(\sum_{n=0}^{N} C_n\zeta^n\right)^3$$

$$=\sum_{n=0}^{3N}\sum_{j=0}^{n}\sum_{k=0}^{j} C_kC_{j-k}C_{n-j}\zeta^n$$

$$=\sum_{n=0}^{3N} V_n\zeta^n \tag{4.4a}$$

where :

$$V_n = \sum_{j=0}^{n}\sum_{k=0}^{j} C_k C_{j-k} C_{n-j} \quad , n=0, 1, 2, \ldots, 3N \tag{4.4b}$$

Substitute equations (4.1), (4.3a), (4.3b) and (4.4a) into the differential equation, one can get:

$$\sum_{n=0}^{N}\{(n+4)(n+3)(n+2)(n+1)C_{n+4} + p(n+2)(n+1)C_{n+2} + k_1 C_n - k_3 V_n\}\zeta^n = 0 \tag{4.5}$$

For non-trivial solution ($\zeta^n \neq 0$), then :

$$(n+4)(n+3)(n+2)(n+1)C_{n+4} + p(n+2)(n+1)C_{n+2} + k_1 C_n - k_3 V_n = 0$$

or, the recurrence formula becomes:

$$C_{n+4} = \frac{k_3 V_n - k_1 C_n - p(n+2)(n+1)C_{n+2}}{(n+4)(n+3)(n+2)(n+1)}, \; n=0,1,2,\ldots,N \tag{4.6}$$

where the non-linear term V_n was given before in equation (4.4b).

Now, in the case of clamped-clamped column, we will try to find a power series form that already achives the boundary conditions at the other side of the column ($\zeta = 1$). The suitable form of this series is:

$$w = (\zeta - 1)^2 \sum_{n=2}^{N} A_n \zeta^n \tag{4.7}$$

This form may be expanded as:

$$w = \sum_{n=2}^{N} A_n \zeta^{n+2} - \sum_{n=2}^{N} 2A_n \zeta^{n+1} + \sum_{n=2}^{N} A_n \zeta^n$$

Shifting the index (n), you will get:

$$w = \sum_{n=4}^{N+2} A_{n-2}\zeta^n - \sum_{n=3}^{N+1} 2A_{n-1}\zeta^n + \sum_{n=2}^{N} A_n\zeta^n$$

$$= -2A_2\zeta^3 + A_2\zeta^2 + A_3\zeta^3 + \sum_{n=4}^{N+2} A_{n-2}\zeta^n - \sum_{n=4}^{N+1} 2A_{n-1}\zeta^n + \sum_{n=4}^{N} A_n\zeta^n$$

$$= A_2\zeta^2 + (A_3 - 2A_2)\zeta^3 + \sum_{n=4}^{N}(A_{n-2} - 2A_{n-1} + A_n)\zeta^n + (A_{N-1} - 2A_N)\zeta^{n+1} + A_N\zeta^{n+2}$$

$$= C_0 + C_1\zeta + C_2\zeta^2 + C_3\zeta^3 + \ldots$$

Equate this series by the truncated series in equation (4.1) to find the relationship between the coefficiants of C's and A's as:

$$\begin{matrix} C_0 = 0 \\ C_1 = 0 \end{matrix} \rightarrow \left.\begin{matrix} A_0 = 0 \\ A_1 = 0 \end{matrix}\right\} \qquad (4.8a)$$

$$C_2 = A_2 \Rightarrow A_2 = C_2 \qquad (4.8b)$$

$$C_3 = A_3 - 2A_2 \Rightarrow A_3 = C_3 + 2A_2 \qquad (4.8c)$$

$C_n = A_{n-2} - 2A_{n-1} + A_n$, for n= 4,5,6,...,N

$\Rightarrow A_n = C_n - A_{n-2} + 2A_{n-1}$, for n=4,5,6,...,N-4 (4.8d)

$$A_{N-1} - 2A_N = 0$$

$$A_N = 0 \qquad (4.8e)$$

$$\Rightarrow A_{N-1} = 0 \qquad (4.8f)$$

Subsitute into equation (4.8d), then:

$$A_N = C_N - A_{N-2} + 2\cancelto{0}{A_{N-1}} = 0 \Rightarrow A_{N-2} = C_N \qquad (4.8g)$$

$A_{N-1}=C_{N-1}-A_{N-3}+2A_{N-2}=0 \Rightarrow A_{N-3}=C_{N-1}+2C_N$ (4.8h)

All the coefficients (A's) are now known.

4.3 CONVERTING THE BUCKLING LOAD EQUATION TO A POWER SERIES FORM.

From section (3.4.3), the buckling load equation is :

$$p=\frac{\int_0^1(\frac{1}{2}w''^2+\frac{1}{2}k_1w^2-\frac{1}{4}k_3w^4)d\zeta}{\int_0^1\frac{1}{2}w'^2d\zeta}$$

In what follows, we will write the previous form of the buckling load in a power series form as :

(a) The term $\int_0^1\frac{1}{2}k_1w^2d\zeta$:

Form equation (4.1), and after it was truncated up to N terms:

$$w=\sum_{n=0}^{N}C_n\zeta^n$$

Then we can write:

$$w^2=\left(\sum_{n=0}^{N}C_n\zeta^n\right)^2$$

$$=\sum_{n=0}^{2N}\sum_{j=0}^{n}C_jC_{n-j}\zeta^n$$

$$=\sum_{n=0}^{2N}Q_n\zeta^n \quad (4.9a)$$

where :

$$Q_n = \sum_{j=0}^{n} C_j C_{n-j} \qquad (4.9b)$$

then the value of the integral becomes:

$$\int_0^1 \frac{1}{2} k_1 w^2 d\zeta = \frac{1}{2} k_1 \int_0^1 \left(\sum_{n=0}^{2N} Q_n \zeta^n \right) d\zeta$$

$$= \frac{1}{2} k_1 \sum_{n=0}^{2N} Q_n \int_0^1 \zeta^n d\zeta = \frac{1}{2} k_1 \sum_{n=0}^{2N} Q_n \left. \frac{\zeta^{n+1}}{n+1} \right|_0^1$$

$$= \frac{1}{2} k_1 \sum_{n=0}^{2N} \frac{Q_n}{n+1} \qquad (4.9c)$$

(b) The term $\int_0^1 \frac{1}{2} w'^2 d\zeta$:

Form equation (4.2b):

$$w' = \sum_{n=1}^{N} n C_n \zeta^{n-1} = \sum_{n=0}^{N-1} (n+1) C_{n+1} \zeta^n$$

Then, w'^2 becomes :

$$w'^2 = \left(\sum_{n=0}^{N-1} (n+1) C_{n+1} \zeta^n \right)^2$$

$$= \sum_{n=0}^{2(N-1)} \sum_{j=0}^{n} (j+1)(n-j+1) C_{j+1} C_{n-j+1} \zeta^n$$

$$= \sum_{n=0}^{2(N-1)} D_n \zeta^n \qquad (4.10a)$$

where :

$$D_n = \sum_{j=0}^{n} (j+1)(n-j+1) C_{j+1} C_{n-j+1} \qquad (4.10b)$$

hence, the value of the integral becomes:

$$\int_0^1 \frac{1}{2} w'^2 d\zeta = \frac{1}{2}\int_0^1 \left(\sum_{n=0}^{2(N-1)} D_n \zeta^n \right) d\zeta$$

$$= \frac{1}{2} \sum_{n=0}^{2(N-1)} D_n \int_0^1 \zeta^n d\zeta = \frac{1}{2} \sum_{n=0}^{2(N-1)} D_n \left. \frac{\zeta^{n+1}}{n+1} \right|_0^1$$

$$= \frac{1}{2} \sum_{n=0}^{2(N-1)} \frac{D_n}{n+1} \tag{4.10c}$$

(c) The term $\int_0^1 \frac{1}{2} w''^2 d\zeta$:

Form equation (4.3a):

$$w'' = \sum_{n=0}^{N-2} (n+2)(n+1) C_{n+2} \zeta^n$$

Then, w''^2 becomes :

$$w''^2 = \left(\sum_{n=0}^{N-2} (n+2)(n+1) C_{n+2} \zeta^n \right)^2$$

$$= \sum_{n=0}^{2(N-2)} \sum_{j=0}^{n} [(j+2)(j+1)][(n-j+2)(n-j+1)] C_{j+2} C_{n-j+2} \zeta^n$$

$$= \sum_{n=0}^{2(N-2)} R_n \zeta^n \tag{4.11a}$$

where :

$$R_n = \sum_{j=0}^{n} [(j+2)(j+1)][(n-j+2)(n-j+1)] C_{j+2} C_{n-j+2} \tag{4.11b}$$

hence, the integral becomes:

$$\int_0^1 \frac{1}{2} w''^2 d\zeta = \frac{1}{2}\int_0^1 \left(\sum_{n=0}^{2(N-2)} R_n \zeta^n \right) d\zeta$$

$$= \frac{1}{2} \sum_{n=0}^{2(N-2)} R_n \int_0^1 \zeta^n d\zeta = \frac{1}{2} \sum_{n=0}^{2(N-2)} R_n \left. \frac{\zeta^{n+1}}{n+1} \right|_0^1$$

$$= \frac{1}{2} \sum_{n=0}^{2(N-2)} \frac{R_n}{n+1} \qquad (4.11c)$$

(d) The term $\int_0^1 \frac{1}{4} k_3 w^4 d\zeta$:

Form equation (4.1):

$$w^4 = \left(\sum_{n=0}^{N} C_n \zeta^n \right)^4$$

$$= \sum_{n=0}^{4N} \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{m=0}^{k} C_m C_{k-m} C_{j-k} C_{n-j} \zeta^n$$

$$= \sum_{n=0}^{4N} F_n \zeta^n \qquad (4.12a)$$

where :

$$F_n = \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{m=0}^{k} C_m C_{k-m} C_{j-k} C_{n-j} \qquad (4.12b)$$

then , the integral will be:

$$\int_0^1 \frac{1}{4} k_3 w^4 d\zeta = \frac{1}{4} k_3 \int_0^1 \left(\sum_{n=0}^{4N} F_n \zeta^n \right) d\zeta$$

$$= \frac{1}{4} k_3 \sum_{n=0}^{4N} F_n \int_0^1 \zeta^n d\zeta = \frac{1}{4} k_3 \sum_{n=0}^{4N} F_n \left. \frac{\zeta^{n+1}}{n+1} \right|_0^1$$

$$= \frac{1}{4} k_3 \sum_{n=0}^{4N} \frac{F_n}{n+1} \qquad (4.12c)$$

Substitute equations (4.9c), (4.10c), (4.11c) and (4.12c) into the equation of p, you will get:

$$p=\frac{\frac{1}{2}k_3\sum_{n=0}^{4N}\frac{F_n}{n+1}-k_1\sum_{n=0}^{2N}\frac{Q_n}{n+1}-\sum_{n=0}^{2(N-2)}\frac{R_n}{n+1}}{\sum_{n=0}^{2(N-1)}\frac{D_n}{n+1}} \qquad (4.13)$$

which represents the non-dimensional buckling load formula in the form of power series.

Chapter Five

RESULTS AND DISCUSSION

5.1 INTRODUCTION

The formulas of both mentioned cases of simply supported and clamped-clamped columns on elastic foundation in trial function solution and the clamped-clamped column on elastic foundation in power series solution are computerized using FORTRAN language and the corresponding computer programs are shown in appendix (A).

In this chapter a parametric study for all parameters that affect the problem will be conducted, these parameters will be valid over ranges which are expected to be physically valid.

The parameters which will be studied are those expected to affect the non-dimensional buckling load, and mode shapes. These parameters are:

1. The linear foundation modulus (k_1), [$k_1=0 \rightarrow k_1=800$].
2. The ratio factor (α), [$\alpha=-3 \rightarrow \alpha=20$].
3. The amplitude of deflection (C), [$C=0.05 \rightarrow C=0.4$].
4. The polynomial degree (N), [$N=20 \rightarrow N=40$].

5.2 ERROR RESULTS FROM USING DIFFERENT FUNCTIONALS.

As it was discussed in chapter three, the exact buckling load (p_2) can be determined by using the functional equation (2.21) with all its terms, while the approximate buckling load (p_3) can be found by using the same functional equation while neglecting the high order terms of bending energy ($\frac{1}{2}EIw'^2w''^2$) and the load potential ($\frac{1}{8}Pw'^4$). The high-order foundation strain energy will remain to be studied. In other words, the approximate buckling load (p_3) was determined by using functional equation (2.22).

Two computer programs shown in Appendix section (A.2), and (A.3) are designed to find (p_2), (p_3) and the percentage error for both simply supported and clamped-clamped cases respectively. The percentage error in buckling load results from using the exact or approximate functional forms will be given as:

$$\%Error = \frac{|p_2 - p_3|}{p_2} \times 100 \qquad (5.1)$$

This percentage error is shown in Figures (5.1) and (5.2) for clamped-clamped and simply supported columns on elastic foundation, respectively, at constant linear elastic foundation modulus $k_1 = 200$. It was shown that the error is negligible and it is within (0-6%) for small deflection problems and small ratio factors (α).

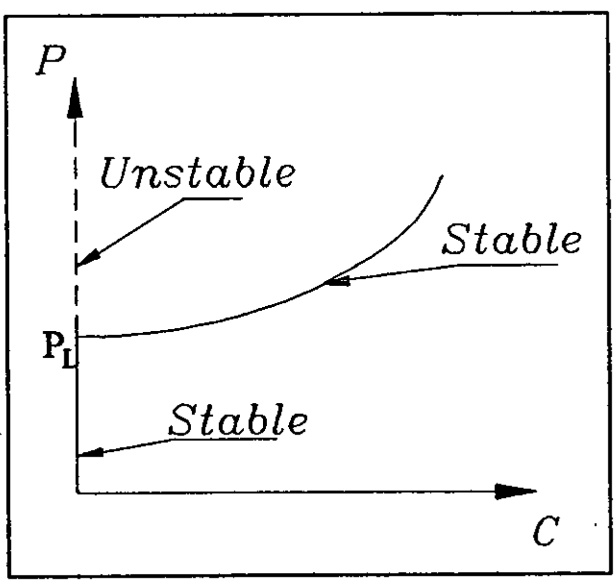
In other words, the percentage error results from using the different previous functionals is reasonable and acceptable for a small hardening and softening small deflection problems. So it is a very good approximation to use the functional equation (2.22) to find the non-linear buckling load in the case of the power series solution as it was shown in chapter four rather than the complicated mathematical form which appears in equation (2.21).

5.3 Parametric study of the problem using trial function method

Structures supported by elastic foundation are quite common in engineering, and the literature on the linear analysis of the beam is extensive. Much less coverage has been given to the non-linear analysis of this class of structures, and, in particular, very little attention has been given to structures supported by a non-linear elastic foundation. So, a complete parametric study addressing the effects of a non-linear elastic parameter on the overall non-linear behavior is unavailable. The present work is an attempt to remedy this shortage of information.

It is well known that the foundation status determines whether the structure is in hardening or in softening. So this will address the effect of linear foundation modulus (k_1), ratio factor (α), amplitude (C) and polynomial order (N) on the overall behavior of columns supported by a foundation that exhibits non-linear hardening and softening.

In order to explain the trends of figures (5.3) through (5.6), let us consider the plot of Figure (5.a) shown below of the load (P) versus the mid point deflection (C), we see that the curve does not start at the origin but at a constant value P_L which represents the classical buckling load, i.e. this constant is the well known Euler critical load of the structure.



Figure(5.a) : The load (P) versus the mid-point deflection (C).

Furthermore, this curve has a horizontal tangent at P_L as expected. This fact can be seen clearly from the initial postbuckling equation (3.9) where we have second order terms (C^2) but no first order terms (C).

Referring to figures (5.3) through (5.6) which represent the variation of the buckling load (p) with the mid point deflection (C) for various values of the non-linear coefficient (α) at specified values of (k_1), we will notice that the increase of the linear foundation modulus (k_1), which is a positive constant, increases the required buckling load of the column (p) when the other parameters are constants (α and C). This happens actually because the strain energy of the foundation increases and the structure becomes more hardened, so a more axial load is required to buckle the column.

Since - as it was mentioned - that the hardened structure needs more axial compressive load than the softened structure to be buckled, then this fact appears clearly in Figures (5.3) through (5.6), for different ratio factors (α) and constant (k_1). In these figures, and at a specified mid point deflection C, it was found that the buckling load increases when the structure becomes more hardened; i.e. when the ratio factor α decreases (from $\alpha = 1.4$ to $\alpha = -1.6$). In other words, it needs more load potential to overcome both bending energy and strain energy of foundation. It is clear from these figures that there is no-effect of the amplitude (C) on the classical buckling load of the linear problem (α=0).

It is clear from Figures (5.7) through (5.10), which represent the mode shapes of deflection for both softening and hardening cases, that the ratio factor (α) has a direct effect on the mode shape. For the hardening case when (α=-0.6), it appears from Figures (5.7) and (5.9) that the buckling load increases as the deflection increases, and this is expected since, as the deflection increases, the non-linear strain energy of the foundation will directly increase and, hence, the overall strain energy of the structure will increase, which means that more load potential is needed to buckle the column. While for the softening case when (α = 0.6) it appears from Figures (5.8) and (5.10) that the buckling load increases as the deflection becomes small; this is because the non-linear strain energy of foundation assists the load potential to overcome the bending energy of the structure. In other words, the non-linear strain energy of foundation plays a role exactly opposite to the role of the linear strain energy of foundation, which will decrease the overall strain energy of the structure and then less buckling load is required.

The linear foundation modulus k_l has a great effect on the resulting mode shape, specially in the case when a power series function is used. This does not mean that it has not any effect when a trial sinusoidal function is used, but it determines only the integer mode shape as it is shown in Figure (5.11). This figure, which represents the mode shape transformation as (k_l) changes, shows that the wave length of the buckled shape decreases as the structure becomes more hardened, as this clearly appears in Equations (5.2) and (5.3).

Similarly, as the effect of (k_l) on the mode shape was discussed the ratio factor (α) also has a great effect on the mode shape. This effect is clearly shown in Figure (5.12),which also represents the mode shape transformation with varying (α). In this case, as the structure becomes more softened, the mode shape becomes more closer to the linear behavior (the wave length of the buckled shape increases), since the effect of α in softening case reduces the effect of the positive linear foundation modulus k_l (i.e. k_{eq} decreases), and hence, the trend of the mode shape becomes, somehow, close to the linear problem. This fact is also emphasized in Figure (5.17) for $\alpha = 20$.

The reason behind the fact that the mode shape number should be integer, i.e. (m = integer), is that it is the only case which satisfies both the differential equation and the boundary conditions at the same time when the trial function of solution is chosen as a sinusoidal wave, i.e. in the case of simply supported column:

$$p = m^2\pi^2 + \frac{k_1}{m^2\pi^2}, \qquad m=1,2,3,...$$

to minimize p, then:

$$\frac{dp}{dm} = 0$$

which yields to:

$$m = \frac{\sqrt[4]{k_1}}{\pi} \qquad (5.2)$$

and in general, for the non-linear problem:

$$m = \frac{\sqrt[4]{k_{eq}}}{\pi} \qquad (5.3)$$

where:

(m) is an integer that comes from approximating the real value to the first integer lower value.

(k_{eq}) is the equivalent foundation modulus which is a function of k_1, α and C, i.e. For simply supported columns

$k_{eq} = k_1(1 - \frac{3}{8}\alpha C^2)$ [See Appendix A].

In this case the mode shape seems to be one or two complete half waves depending on the mode shape number (m), and it does not appear anyhow in between.

5.4 PARAMETRIC STUDY OF THE PROBLEM USING POWER SERIES METHOD

In the case of power series solution, it clearly appears in Figures (5.13) through (5.16), which represent the mode shapes for different values of linear foundation modulus (k_1), that the mode shapes transferred from the first mode to the second mode and scans all the shapes in between at different values of k_1. In other word, since the mode shape in between satisfies both differential equation and boundary conditions, then the solution exists.

To be more specific, Figure (5.13) shows the transformation of the mode shape from the first mode (k_1=0) up to the second mode ($k_1 \approx 700$). The classical buckling load increases when - as it was expected - the deflection is increased, since the structure becomes harder and harder. It should be pointed here that the classical buckling load is not affected by the amplitude since the problem is linear.

Figure (5.14) shows the slope of the deflected line of elastica at various points along the column. This figure insures that the boundary conditions are satisfied. It also shows that the slope of the column at its fixed ends increases as the linear foundation modulus k_1 increases, and hence the buckling load does.

The explanation of the behavior of Figures (5.15) and (5.16) is similar to that of Figures (5.13) and (5.14), respectively, except that they represent the case of hardening structure. In this case, at each value of linear foundation modulus (k_1), the solution exists and the boundary condition are satisfied. In this case the solution is gradually transformed to the second mode as the structure becomes more hardened and, hence, the buckling load increases.

The effect of the ratio factor α on the mode shapes is clearly presented in Figures (5.17) through (5.20) in the case of the power series solution. It was shown that as the structure becomes more hardened, i.e. (α decreases from α=20 to α=0), then it needs more axial compressive load to be buckled. That is because the equivalent overall strain energy of foundation will increase. From these figures, it is clear that the mode shape is transformed to the higher modes as the structures becomes more hardened and, on the other hand, it seems to look like the linear behavior when the structure becomes very softened ($\alpha = 20$).

This fact has been discussed in Figures (5.11) and (5.12). Here, it should be pointed that the mode shape transformation in Figure (5.19) when k_1=600 is more clear than that in Figure (5.17) when k_1=100, since the structure is more hardened. The slopes represented in Figures (5.18) and (5.20) are shown to insure that all the boundary conditions have been satisfied.

As we found previously in the trial function solution, the same was found in the power series method. Figure (5.21) which represents load-deflection diagram for different values of ratio factor (α), shows that the required buckling load increases as the column becomes more hardened for a specified value of deflection (C), i.e. when the ratio factor (α) decreases from $\alpha = 3$ to $\alpha = 0$. This is because - as it was previously explained - the total bending energy and strain foundation energy will increase when the ratio factor (α) decreases, which requires, for sure, more load potential to overcome those energies, and hence, more compressive buckling load.

What was obtained in the trial function results, is expected to happen in the power series method in Figure (5.22). This figure reflects the fact that the structure needs more axial applied load to be buckled when it becomes more hardened. The classical buckling load (p) in the case when no foundation exists (k_1=0) appears to be equal to 39.2.

Figures (5.23) and (5.24) emphasize the fact that appeared in Figures (5.3) through (5.6) and show - as it was expected - how the load is decaying as the structure becomes more softened for various values of the linear foundation modulus (k_1). In both figures, it is clearly that the buckling load level in the case of k_1=600 is higher then the level in the case of k_1=100, since the structure is more hardened.

5.5 COMPARISON BETWEEN SIMPLY SUPPORTED AND CLAMPED-CLAMPED COLUMNS USING TRIAL FUNCTION METHOD

Figures (5.25) through (5.27) present the results of both simply supported and clamped-clamped columns on elastic foundation on the same graph.

These figures show that the clamped-clamped column needs more axial compressive load than the simply supported column to be buckled, and since its ends are fixed then the deflected shape is expected to be at a level less than that of the simply supported column with horizontal slope at its clamped ends.

5.6 EFFECT OF THE POLYNOMIAL DEGREE (N)

The approximating function in the power series method shows an erratic and unpredictable solution before a certain value of polynomial order (N). For each order, there are an infinite number of buckling loads and hence infinite number of mode shapes. The minimum one is the only one which was chosen as the buckling load of the structure.

Figures (5.28) through (5.31) show the effect of the approximating function order (N) on the estimated non-dimensional buckling load (p) for a clamped-clamped column on elastic foundation for various values of (k_1) and (α). From these figures, the study of the solution convergence shows that the estimated non-dimensional buckling load encounters some error at a low polynomial degree (N). By increasing the approximating function order, the error in the estimated non-dimensional buckling load decreases. After a certain polynomial degree (N), the non-dimensional buckling load will reach a steady value, that value is the exact value of the non-dimensional buckling load.

The running time of the power series program in Appendix section (A.4) depends on the problem considered, the approximating function order (N), the solution convergence tolerance and the speed of the computer used. Depending on these factors the running time varies from 5 minutes to more than half an hour.

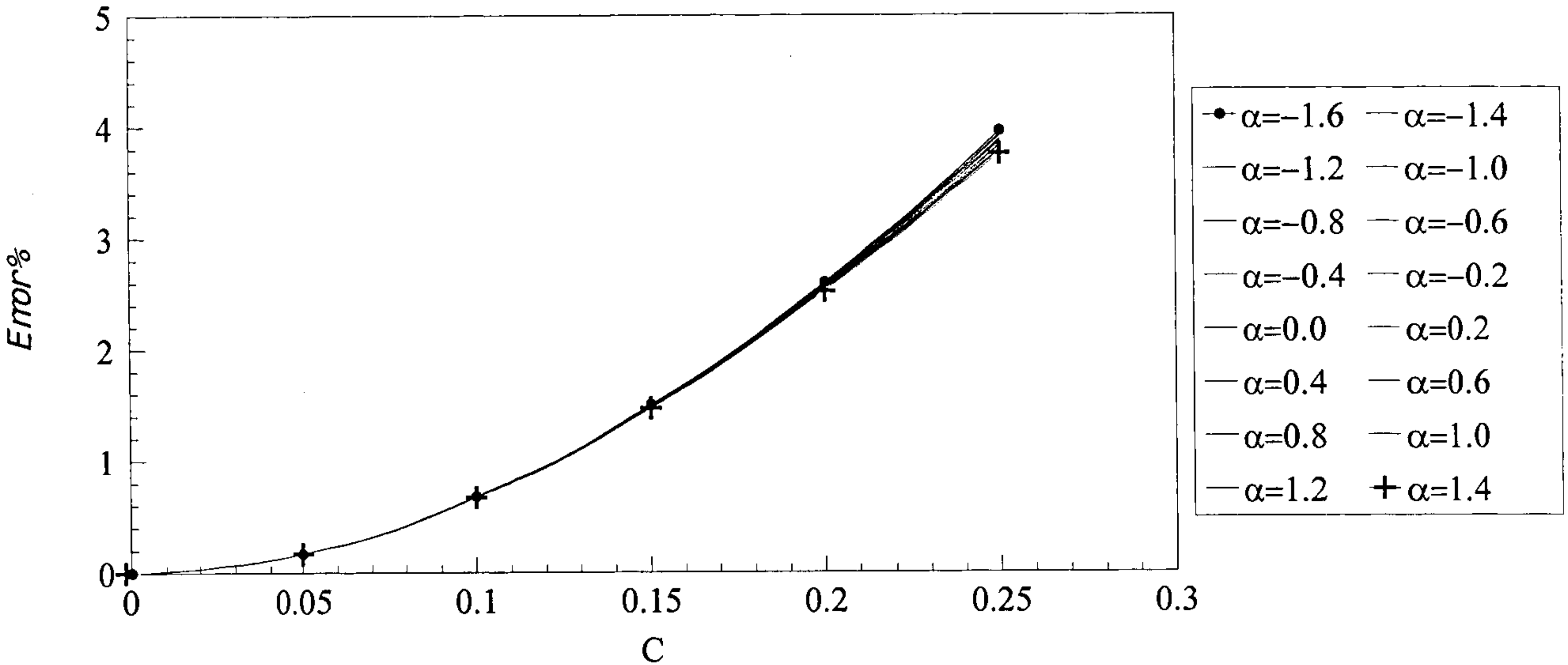


Figure (5.1) : Percentage error for buckling load p versus the mid point deflection due to using different functional forms for clamped-clamped column on elastic foundation with k1=200, using the trial function method.

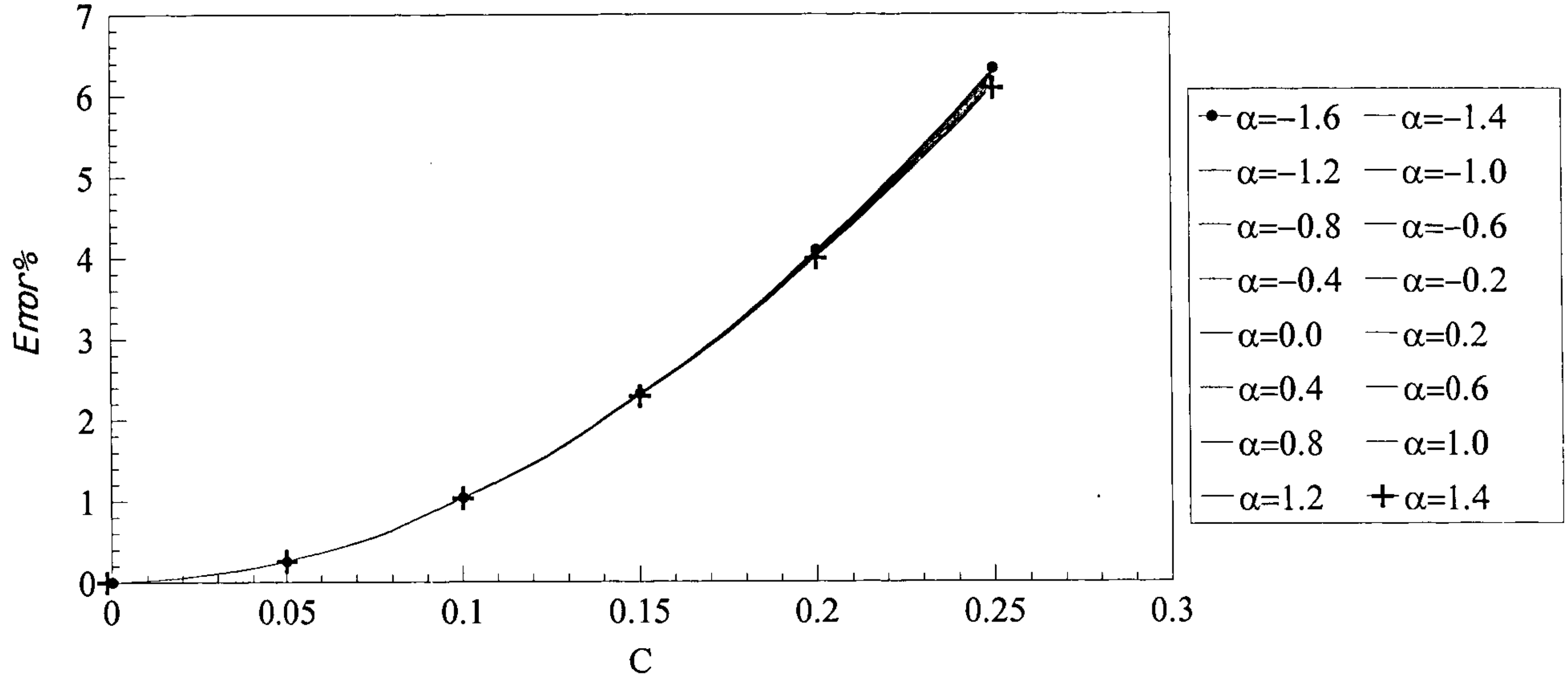


Figure (5.2) : Percentage error for buckling load p versus the mid point deflection due to using different functional forms for simply supported column on elastic foundation with k1=200, using the trial function method.

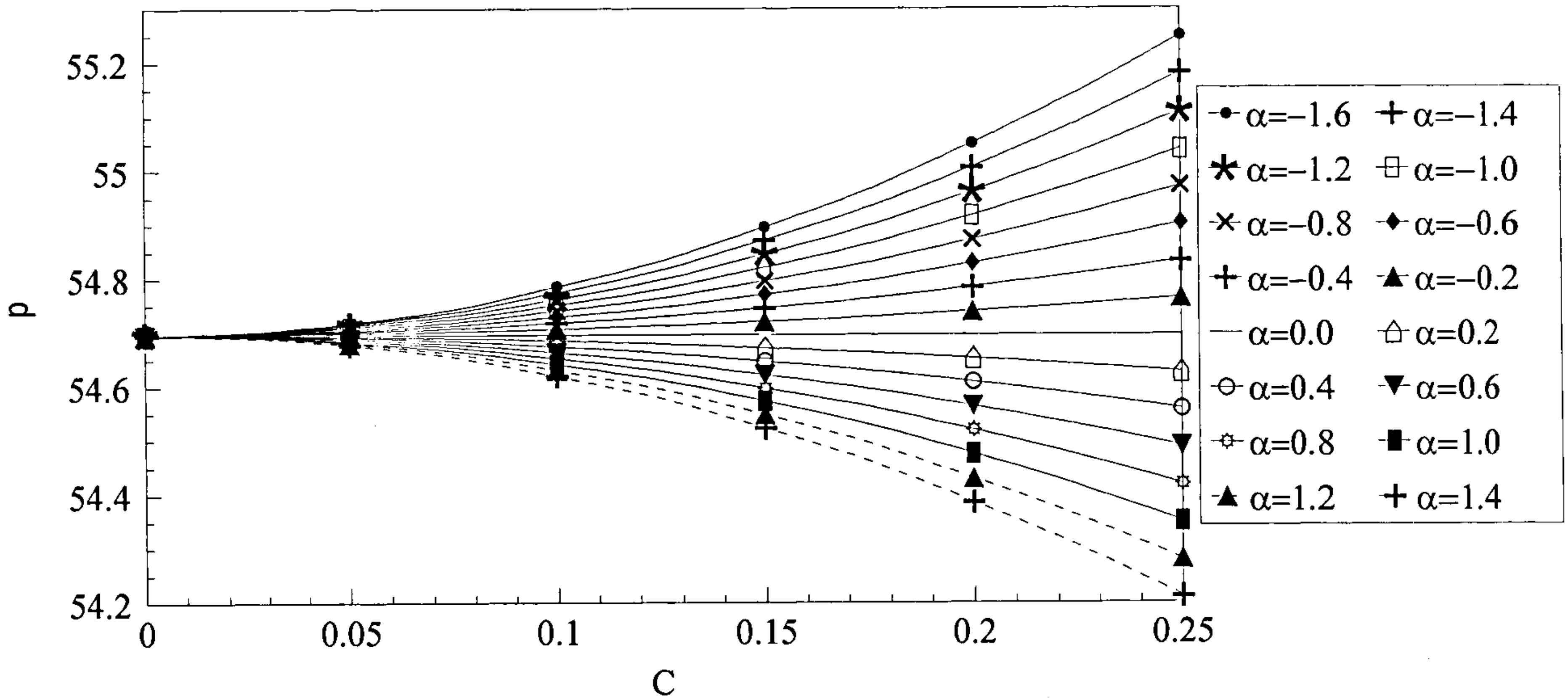


Figure (5.3) : Buckling load p versus midpoint deflection C value for clamped-clamped column on elastic foundation for k1=200, and various values of ratio factor α, using the trial function method.

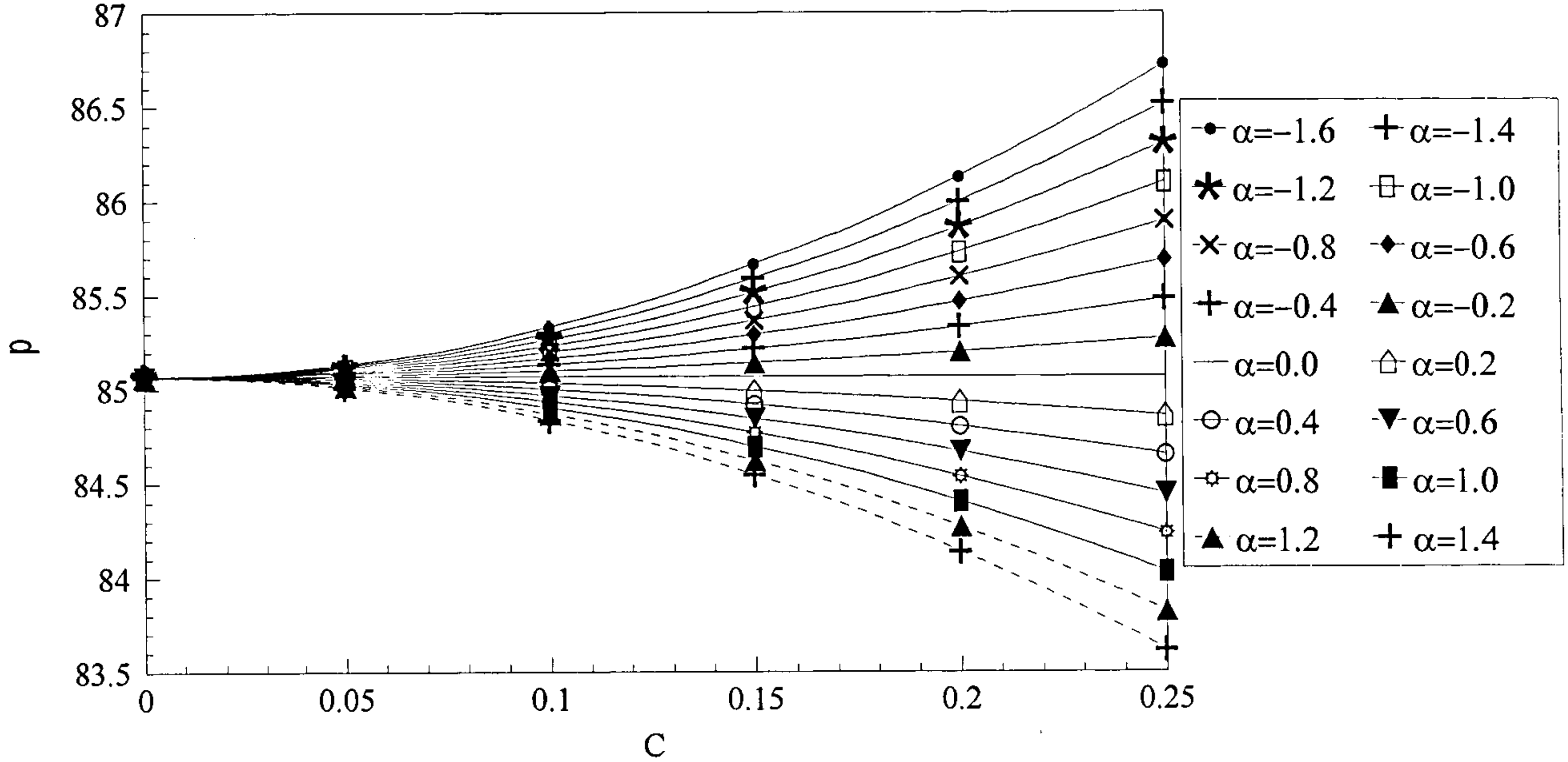


Figure (5.4) : Buckling load p versus midpoint deflection C value for clamped-clamped column on elastic foundation for k1=600, and various values of ratio factor α, using the trial function method.

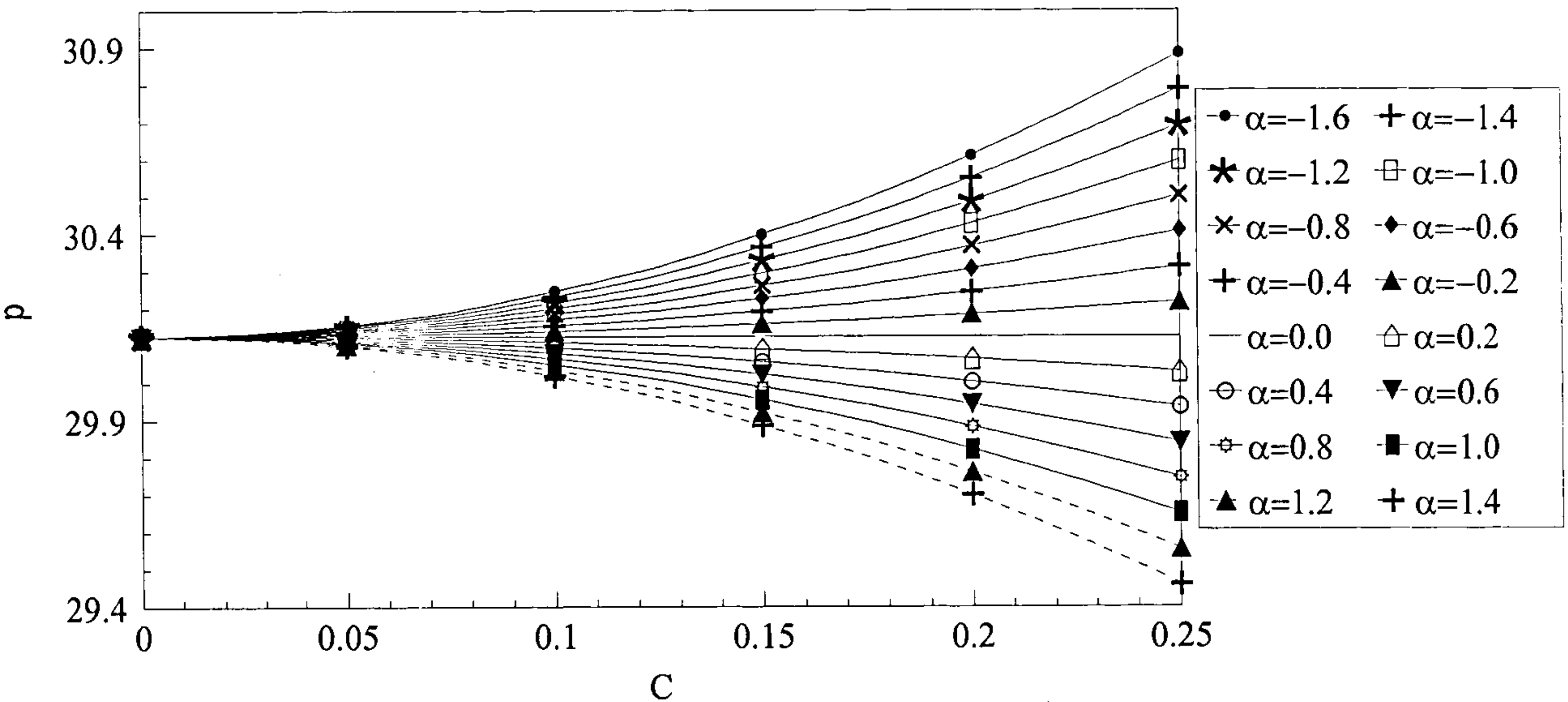


Figure (5.5) : Buckling load p versus midpoint deflection C value for simply supported column on elastic foundation for k1=200, and various values of ratio factor α, using the trial function method.

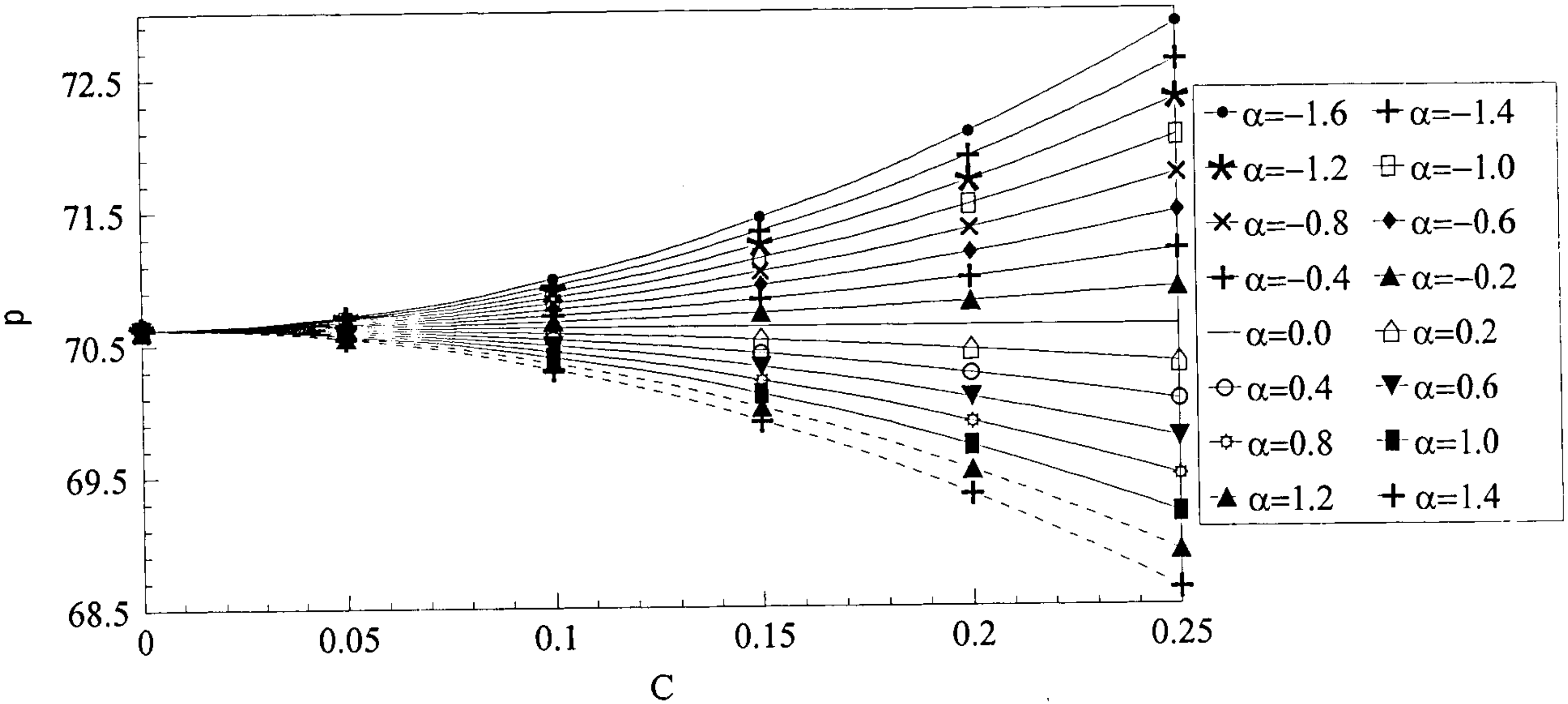


Figure (5.6) : Buckling load p versus midpoint deflection C value for simply supported column on elastic foundation for k1=600, and various values of ratio factor α, using the trial function method.

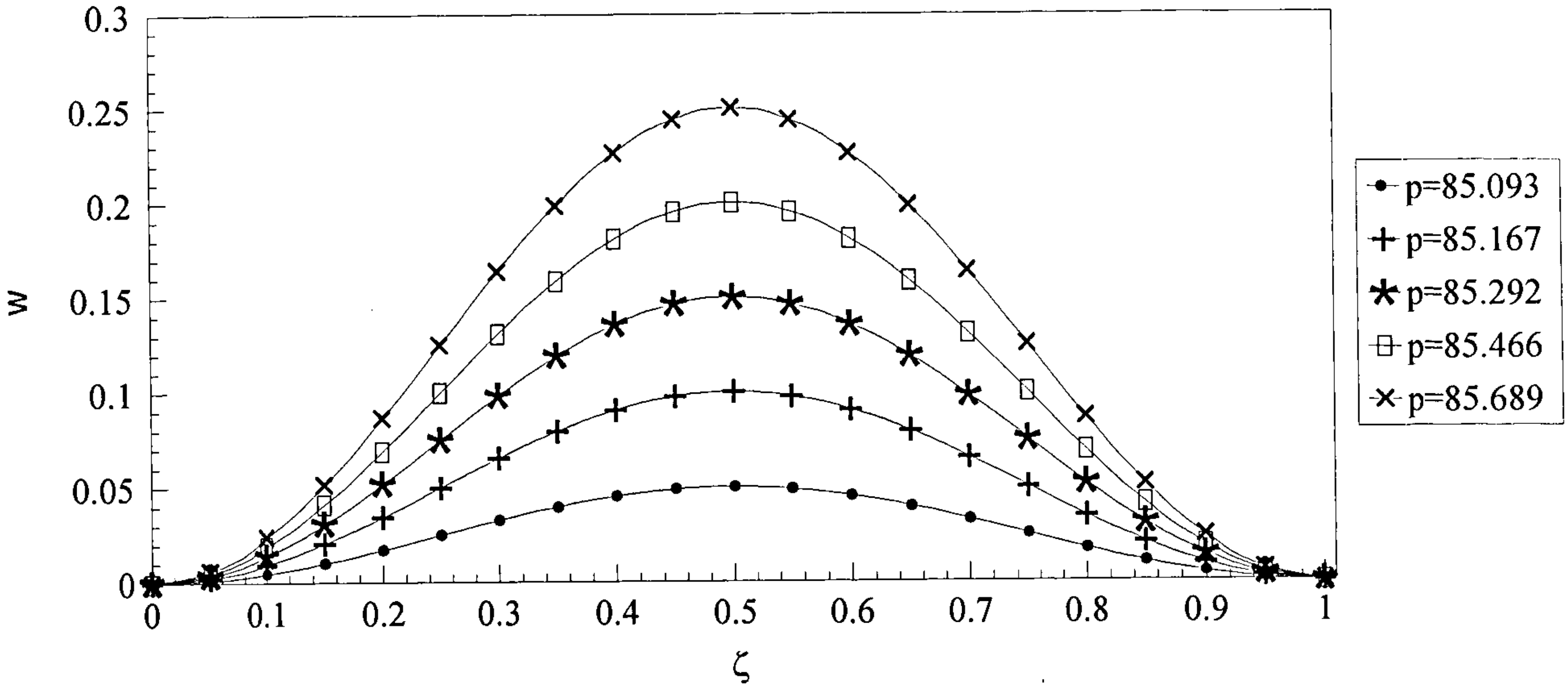


Figure (5.7) : Mode shapes of deflection for clamped-clamped column with k1=600, α=-0.6 and various midpoint deflections, using the trial function method.

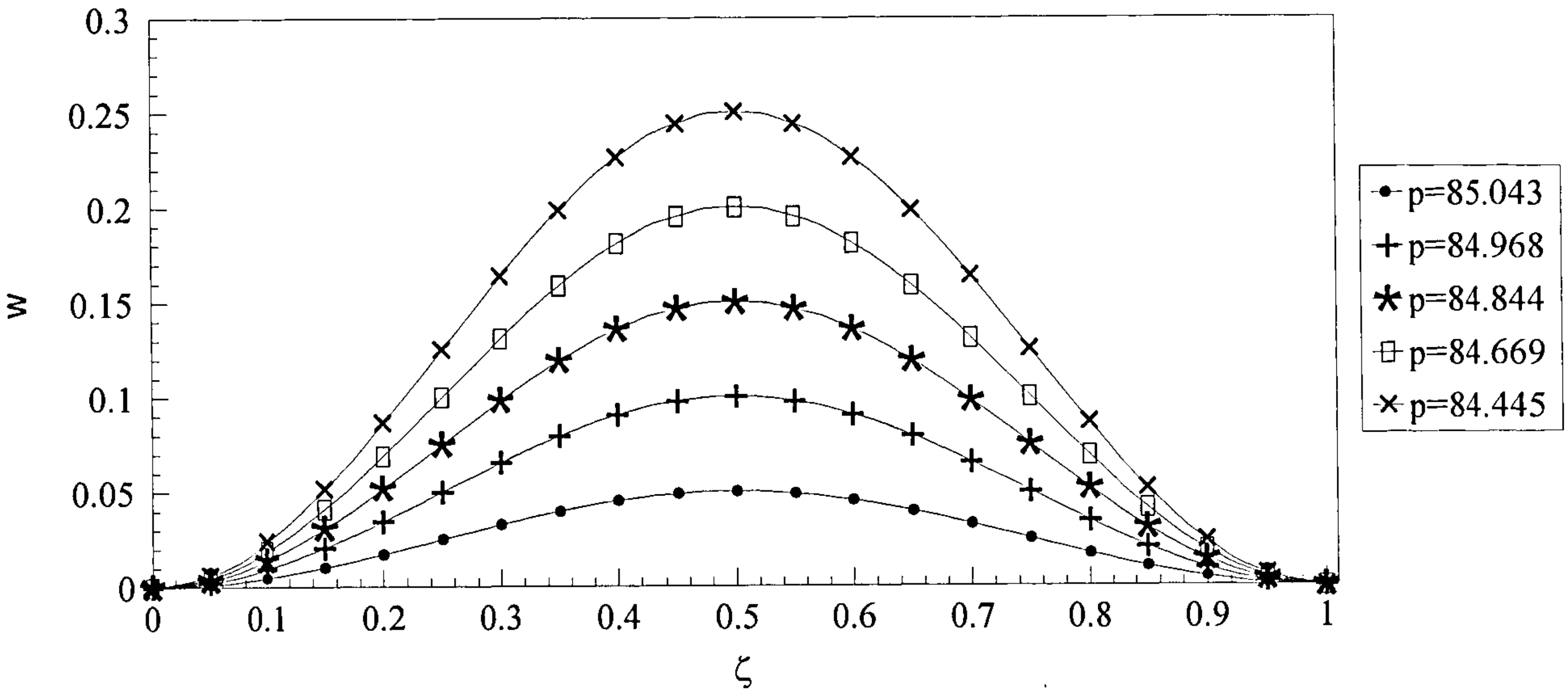


Figure (5.8) : Mode shapes of deflection for clamped-clamped column with k1=600, α=0.6 and various midpoint deflections, using the trial function method.

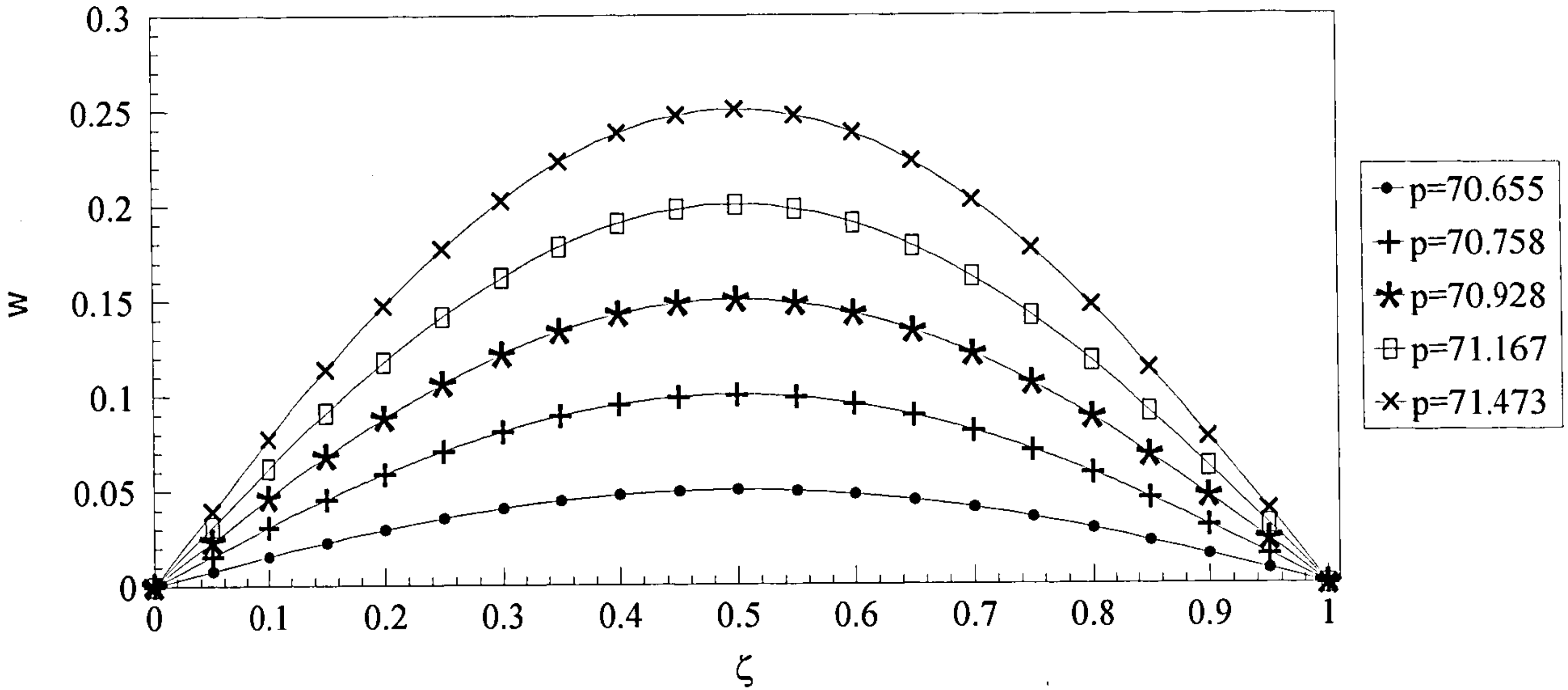


Figure (5.9) : Mode shapes of deflection for simply supported column with k1=600, α=-0.6 and various midpoint deflections, using the trial function method.

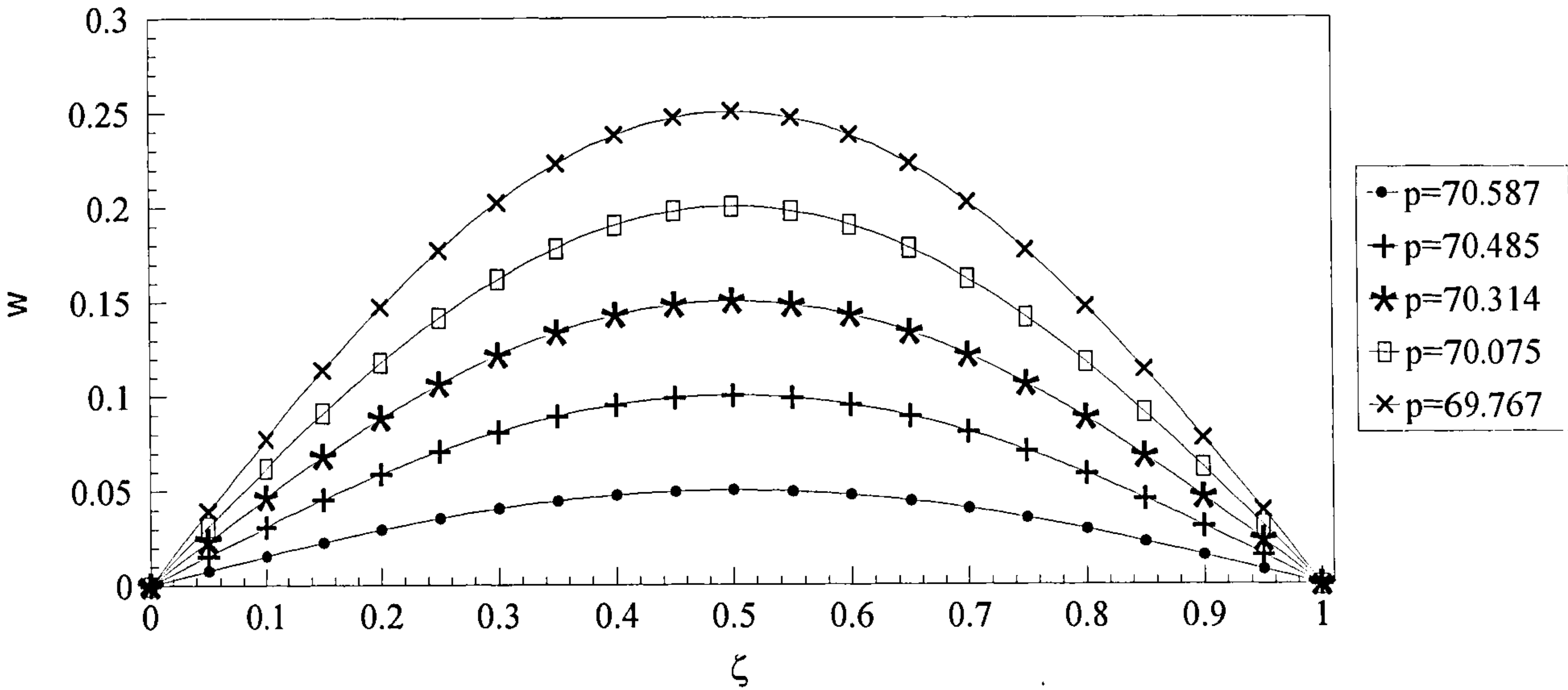


Figure (5.10) : Mode shapes of deflection for simply supported column with k1=600, α=0.6 and various midpoint deflections, using the trial function method.

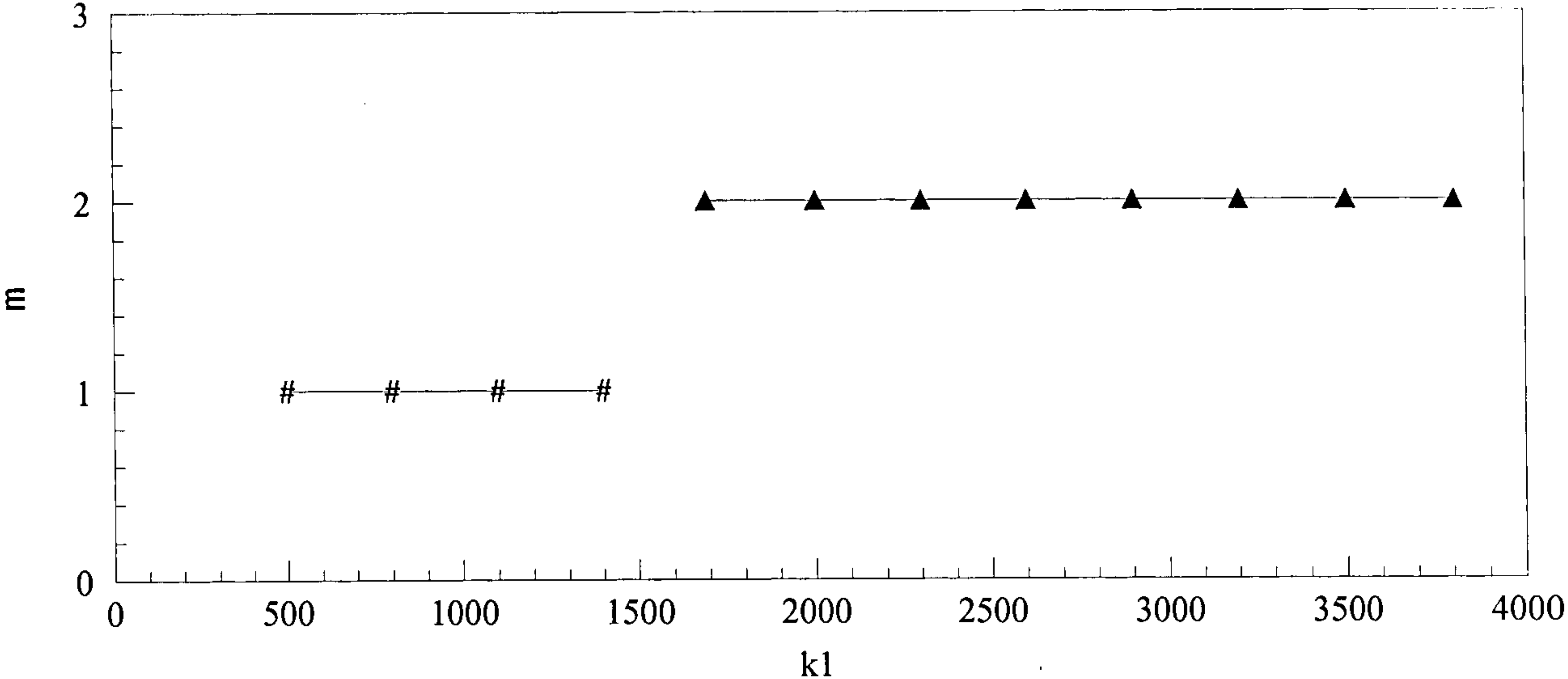


Figure (5.11) : Mode shape number m versus linear foundation modulus k1 for simply supported column on elastic foundation for ratio factor α=1.9 , and mid point deflection C = 0.1 using the trial function method.

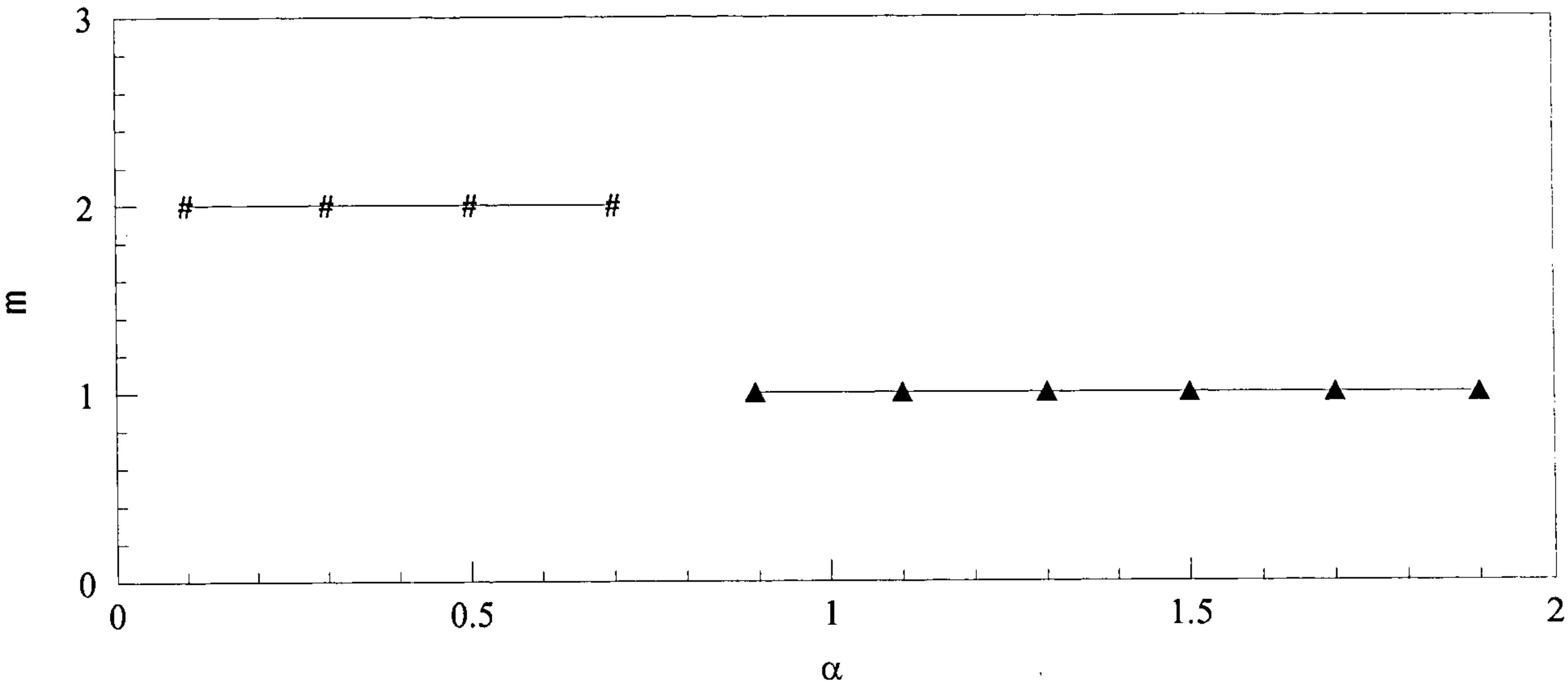


Figure (5.12) : Mode shape number m versus ratio factor α for simply supported column on elastic foundation for linear foundation modulus k1=1700, and mid point deflection C = 0.5 using the trial function method.

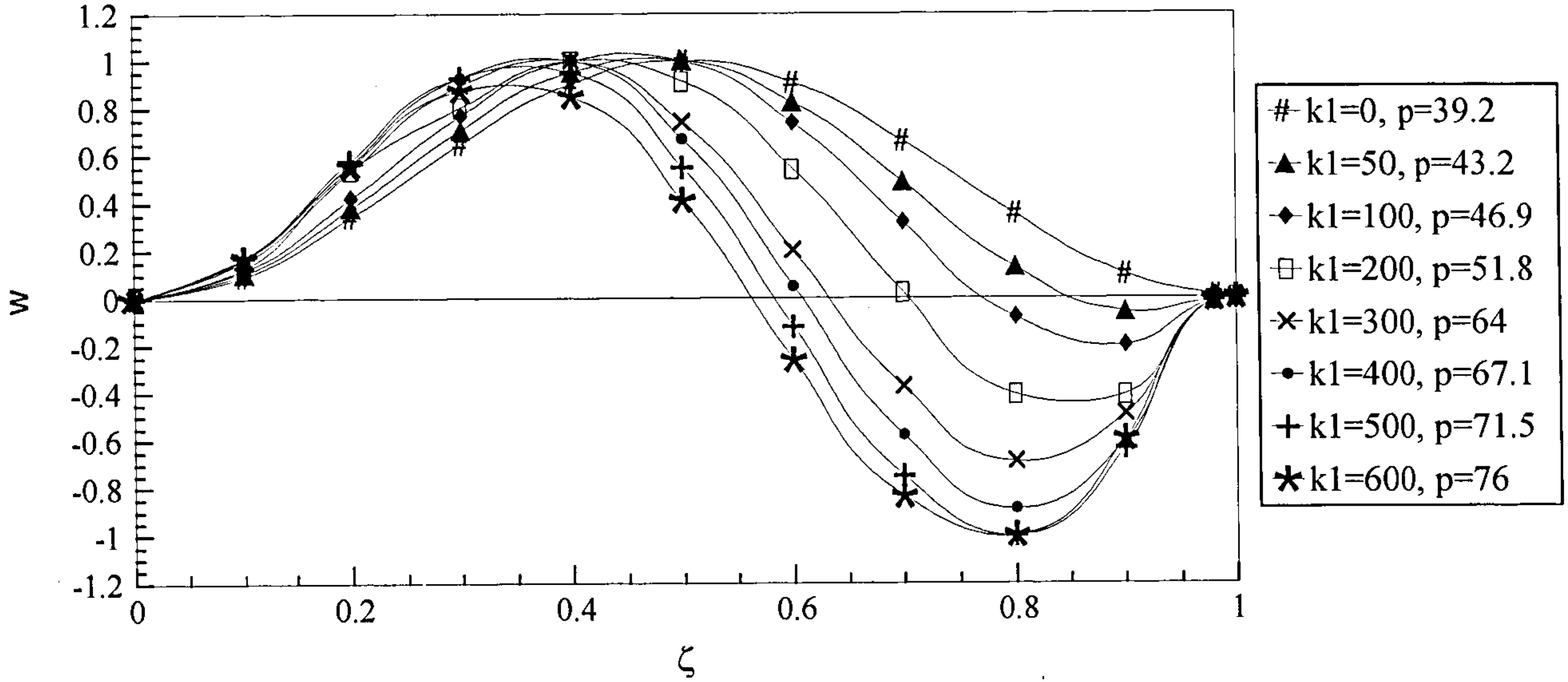


Figure (5.13) : Mode shapes of normalized deflection for clamped-clamped column on elastic foundation for N=20, $\alpha=0$, and starting C value = 0.2 for various values of linear foundation modulus k1, using the power series method.

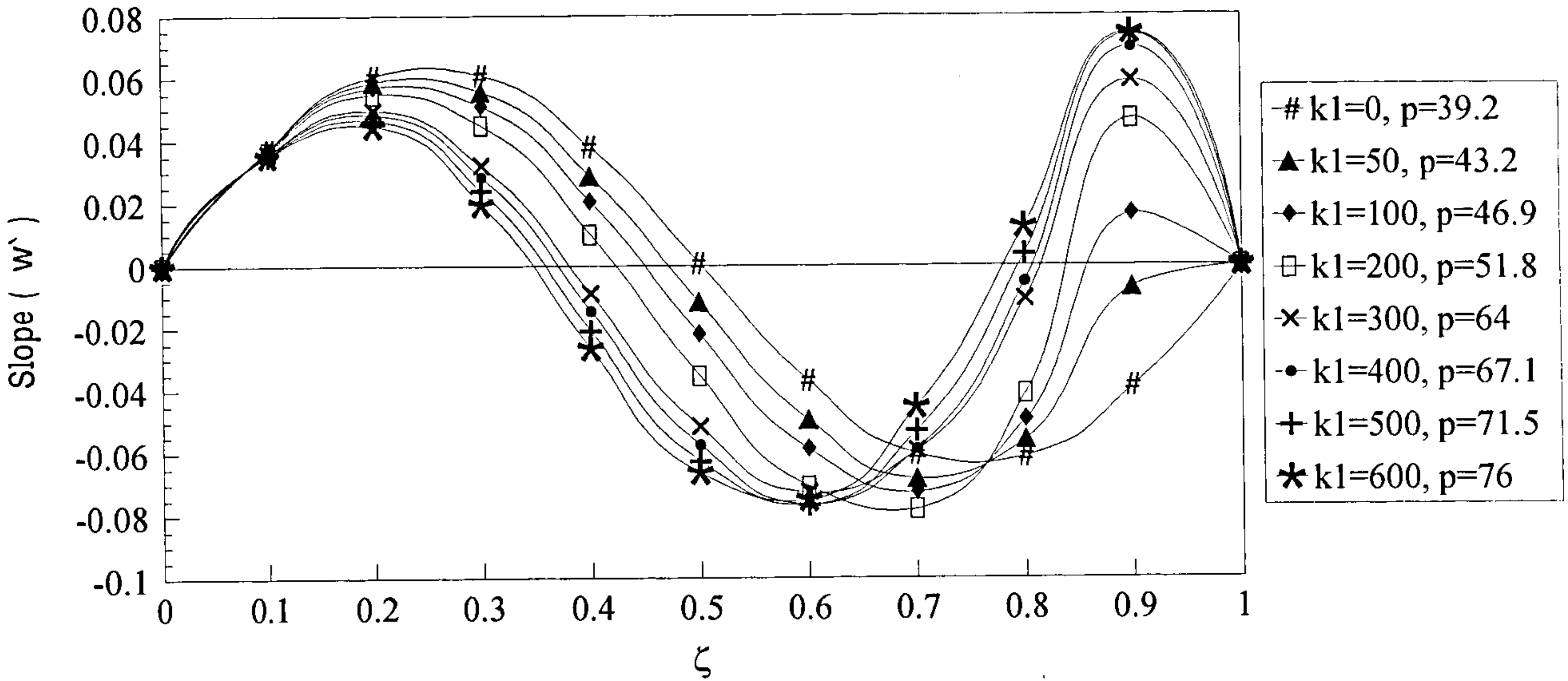


Figure (5.14) : Slope of the deflected shape for clamped-clamped column on elastic foundation for N=20, α=0, and starting C value = 0.2 for various values of linear foundation modulus k1, using the power series method.

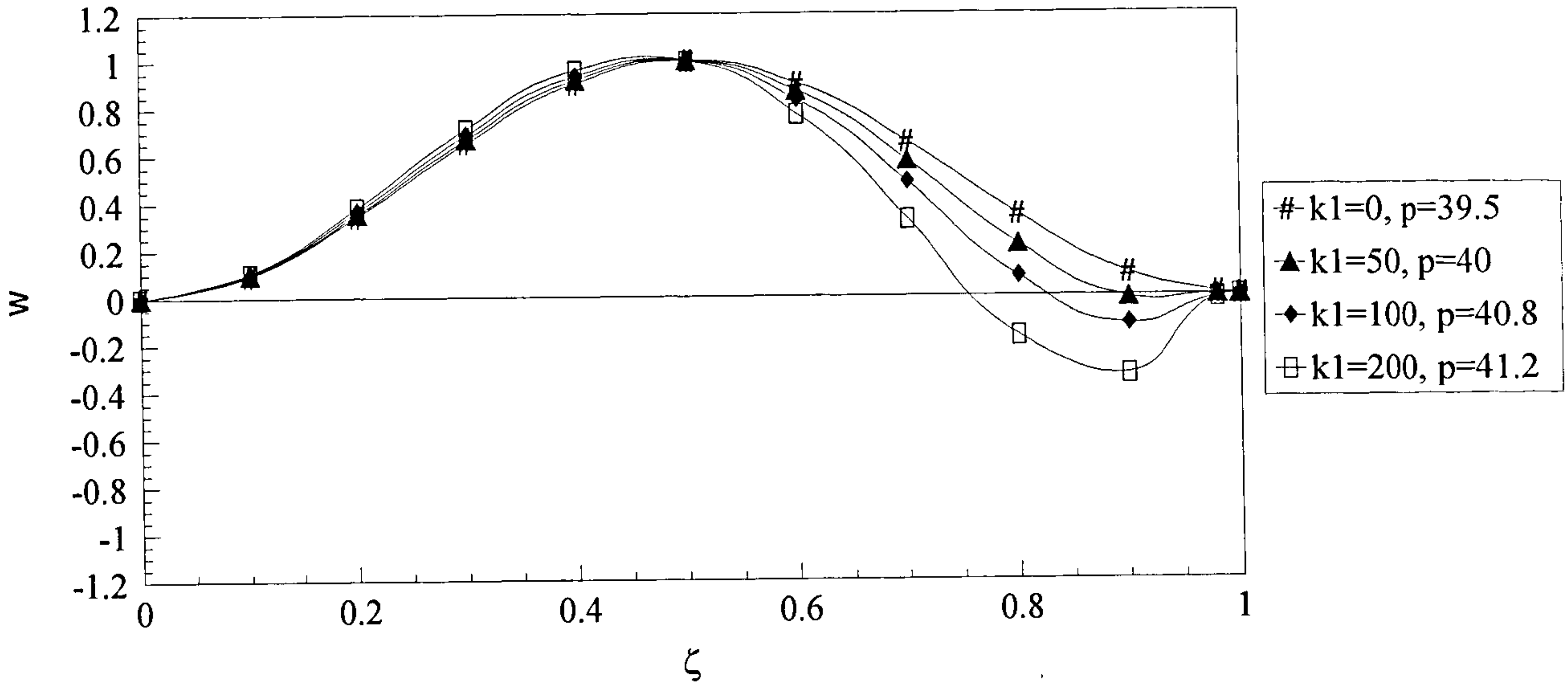


Figure (5.15) : Mode shapes of normalized deflection for clamped-clamped column on elastic foundation for N=20, α=-0.6, and starting C value = 0.2 for various values of linear foundation modulus k1, using the power series method.

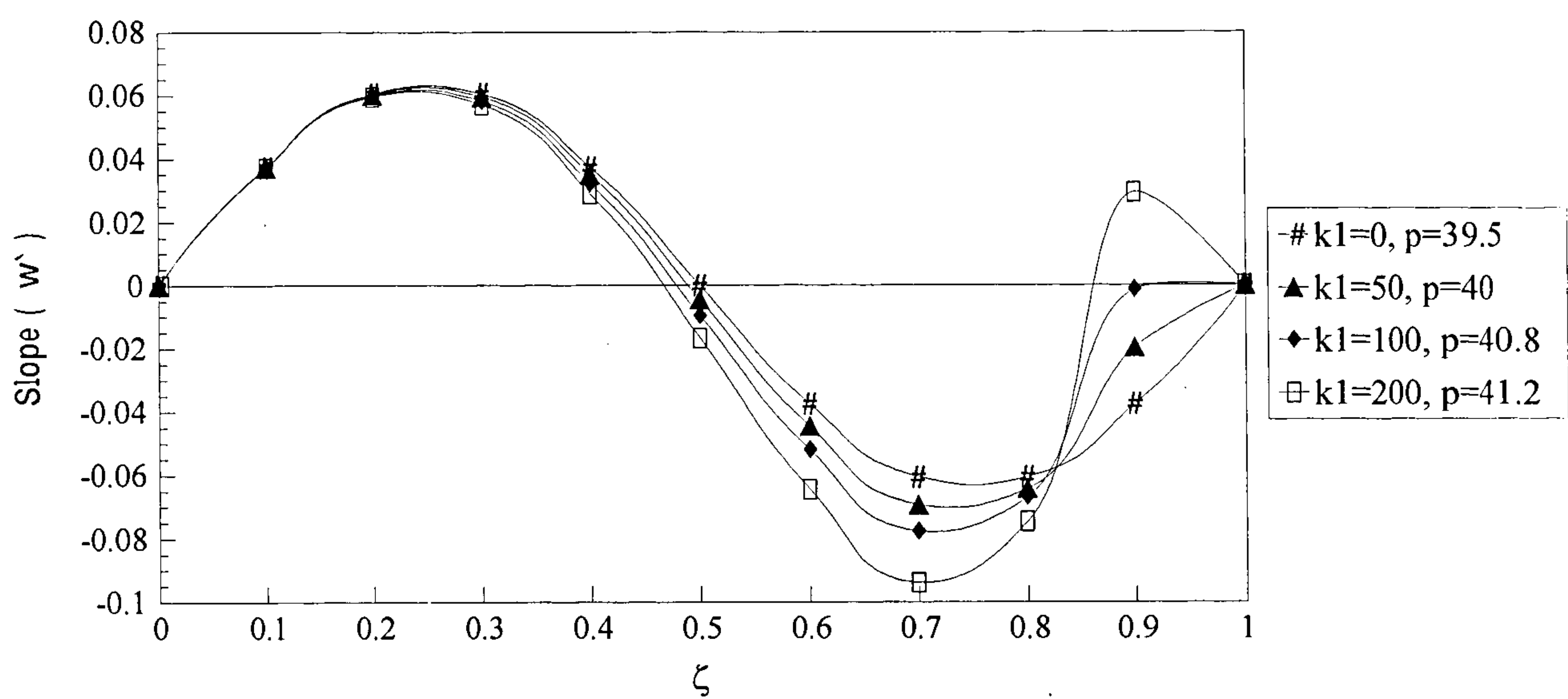


Figure (5.16) : Slope of the deflected shape for clamped-clamped column on elastic foundation for N=20, α=-0.6, and starting C value = 0.2 for various values of linear foundation modulus k1, using the power series method.

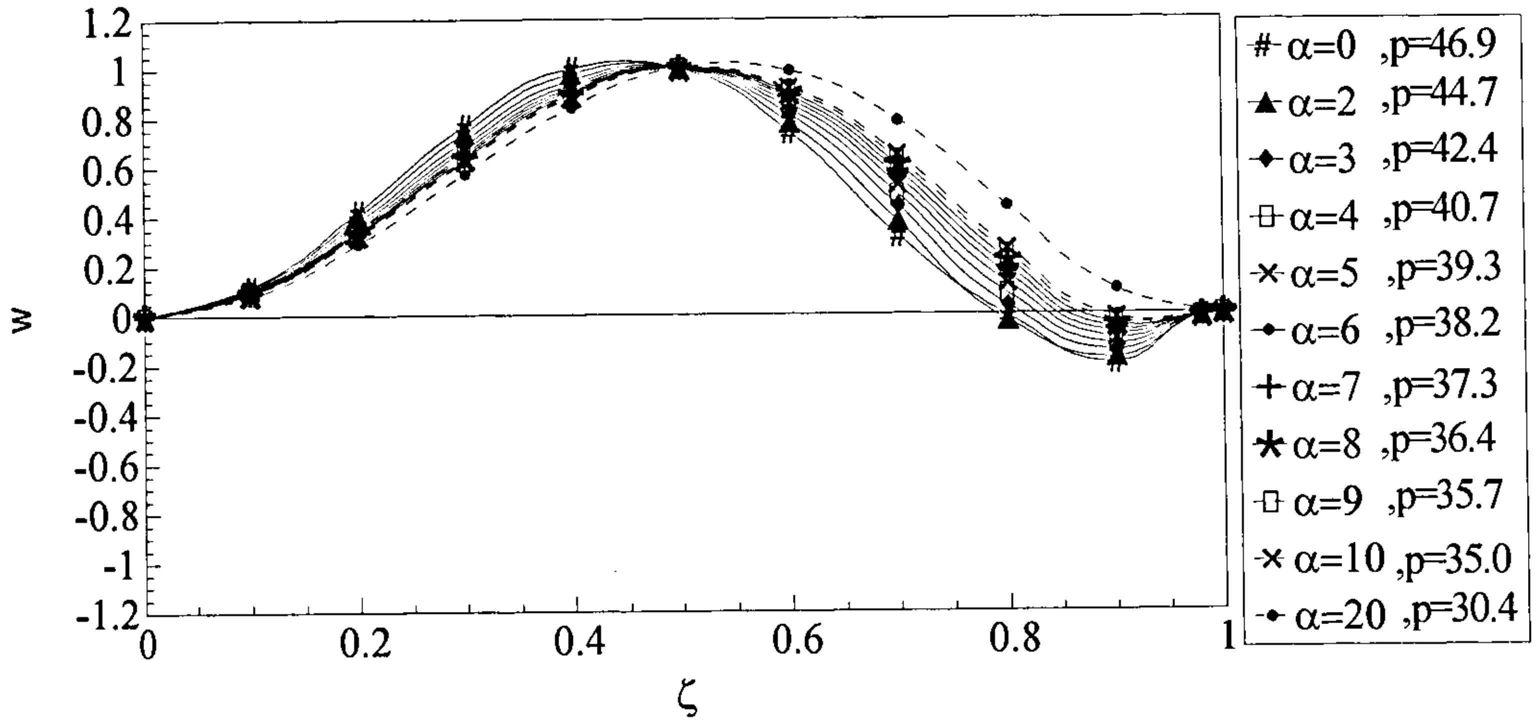


Figure (5.17) : Mode shapes of normalized deflection for clamped-clamped column on elastic foundation for N=20, k1=100, and starting C value = 0.2 for various values of ratio factor α, using the power series method.

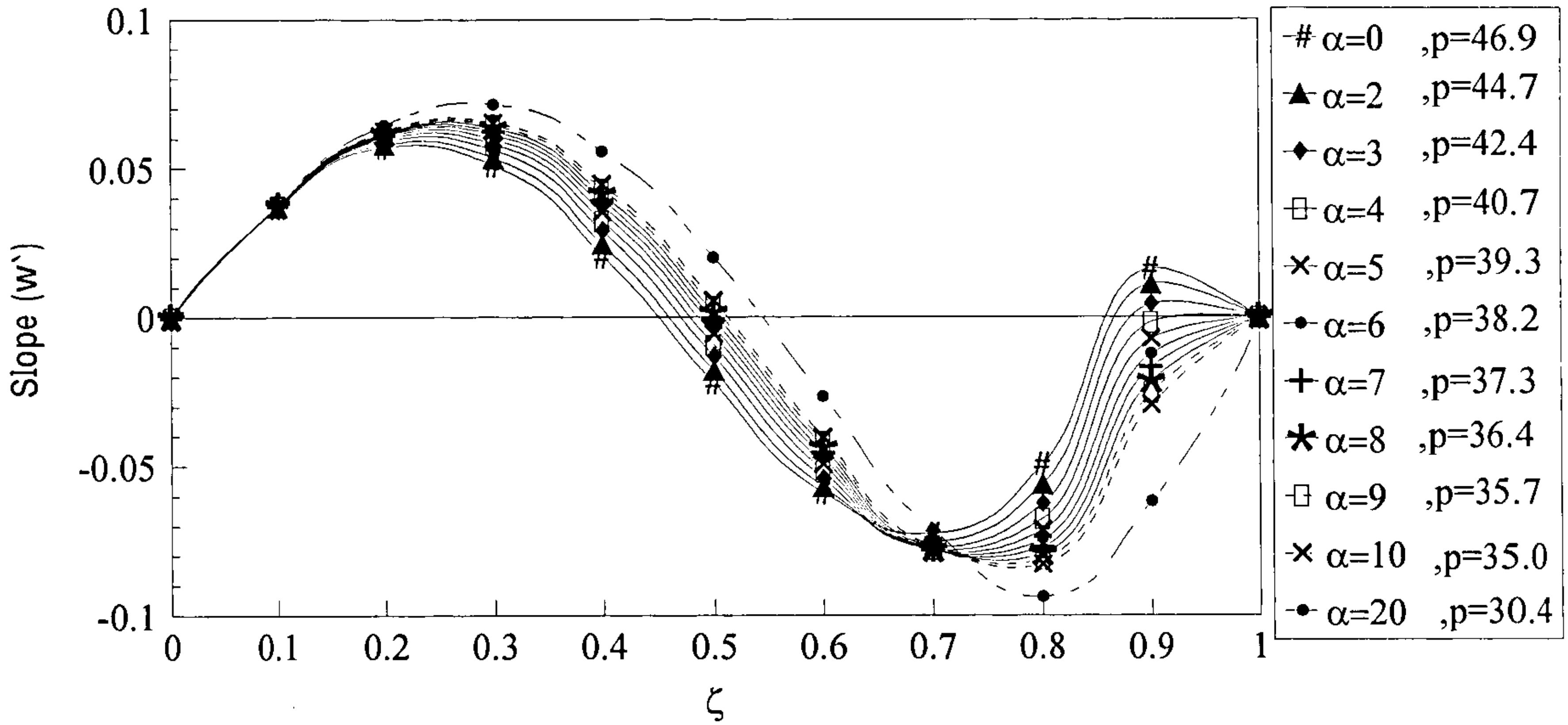


Figure (5.18) : Slope of the deflected shape for clamped-clamped column on elastic foundation for N=20, k1=100, and starting C value = 0.2 for various values of ratio factor α, using the power series method.

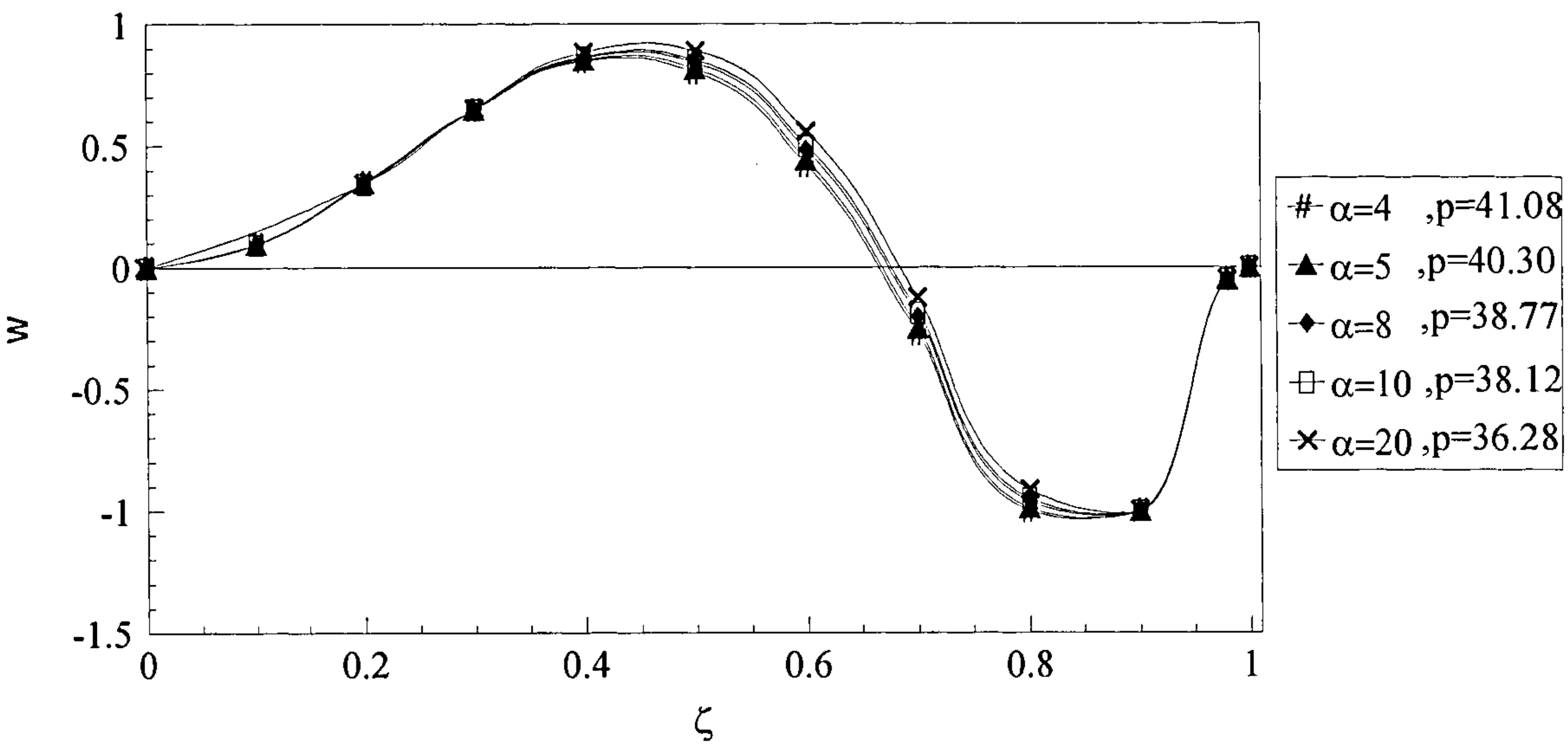


Figure (5.19) : Mode shapes of normalized deflection for clamped-clamped column on elastic foundation for N=20, k1=600, and starting C value = 0.2 for various values of ratio factor α, using the power series method.

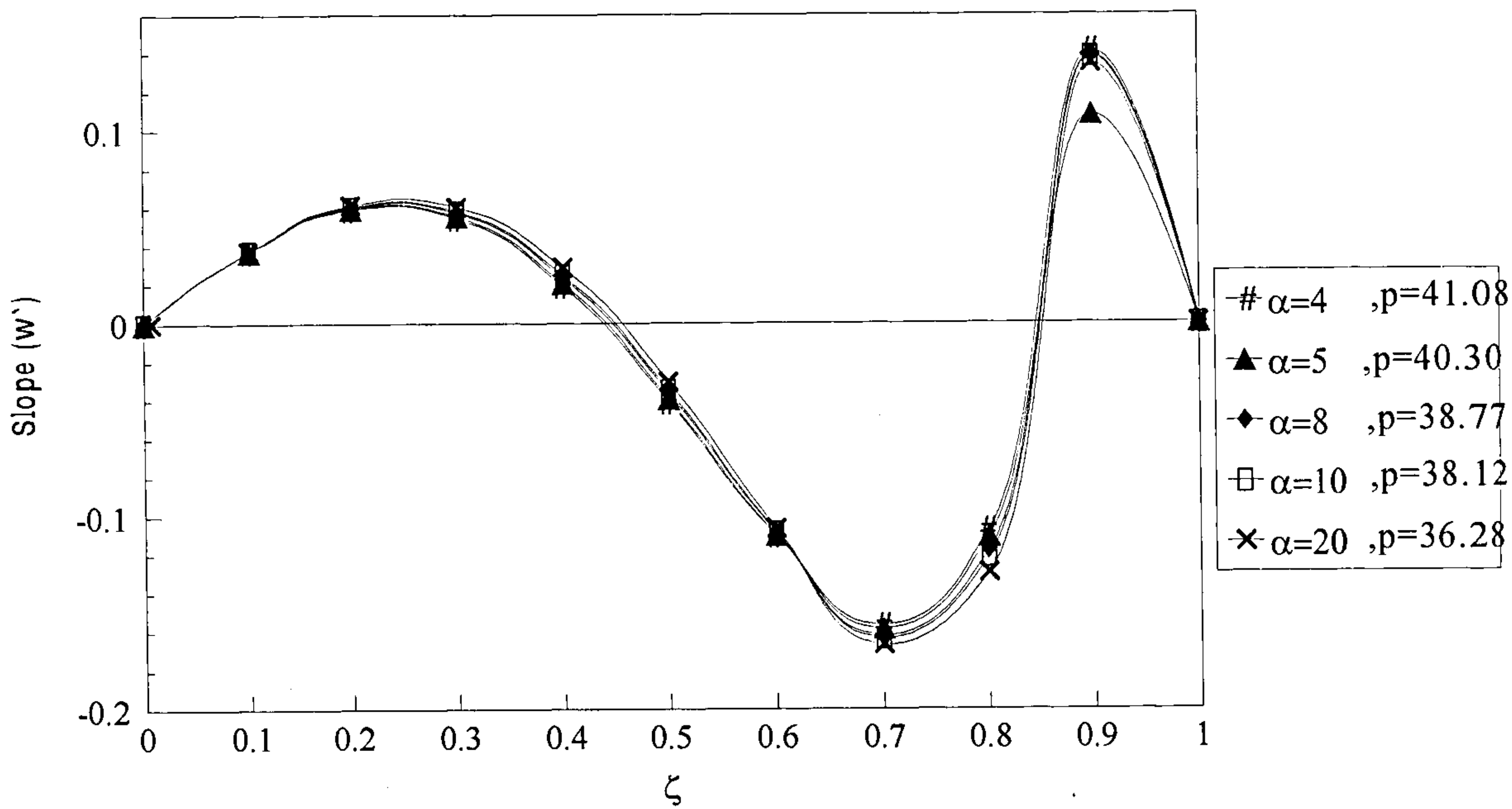


Figure (5.20) : Slope of the deflected shape for clamped-clamped column on elastic foundation for N=20, k1=600, and starting C value = 0.2 for various values of ratio factor α, using the power series method.

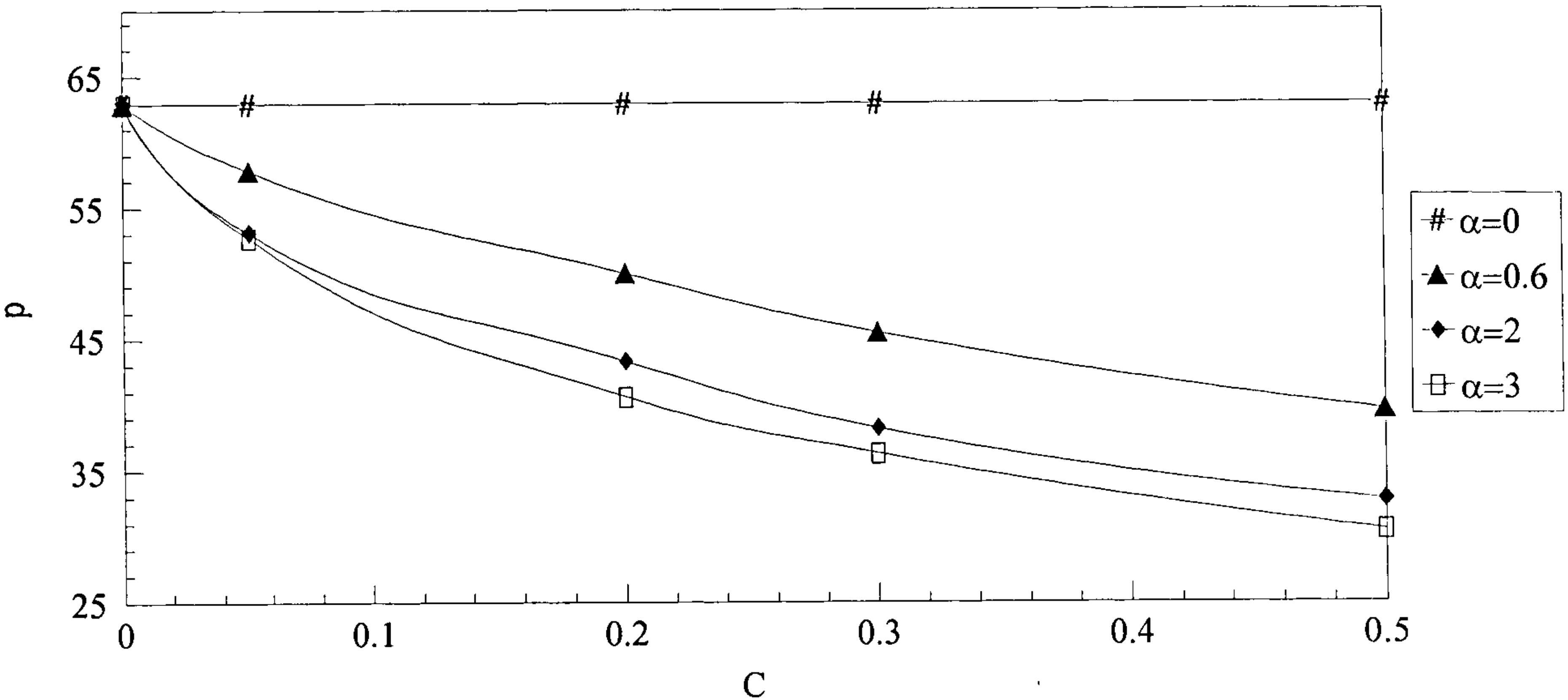


Figure (5.21) : Buckling load p versus starting C value for clamped-clamped column on elastic foundation for N=20, k1=200 for various values of ratio factor α, using the power series method.

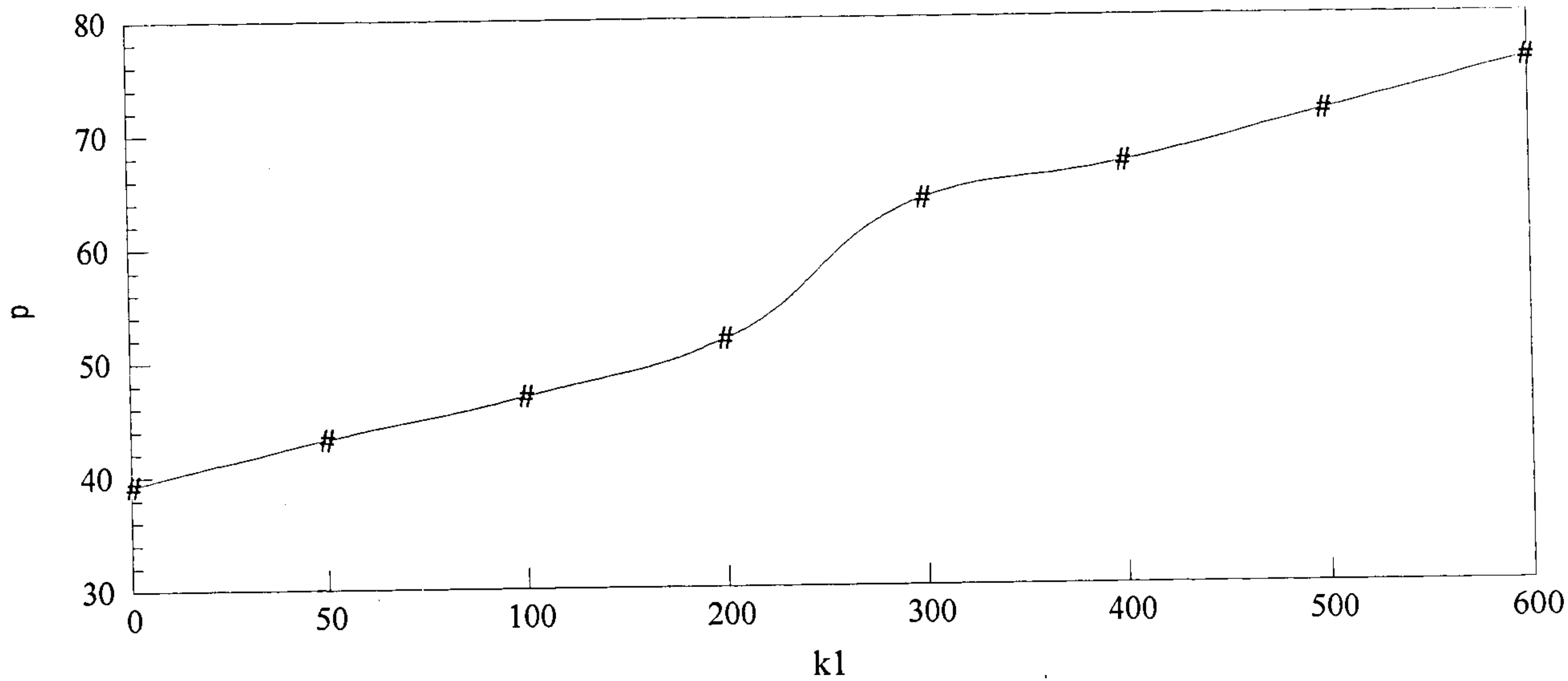


Figure (5.22) : Buckling load p versus linear foundation modulus k1 for clamped-clamped column on elastic foundation for N=20, α=0, and starting C value = 0.2, using the power series method.

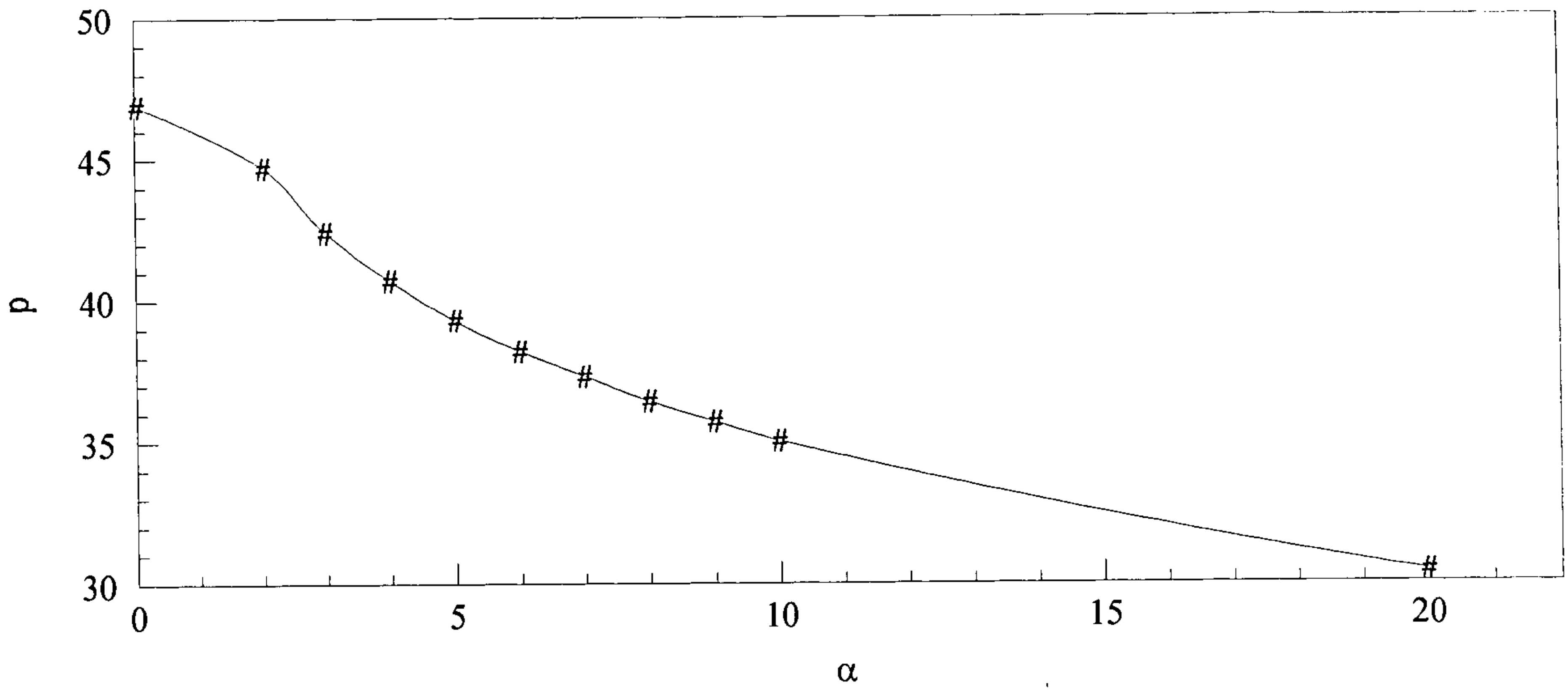


Figure (5.23) : Buckling load p versus ratio factor α for clamped-clamped column on elastic foundation with N=20, k1=100 , and starting C value = 0.2, using the power series method.

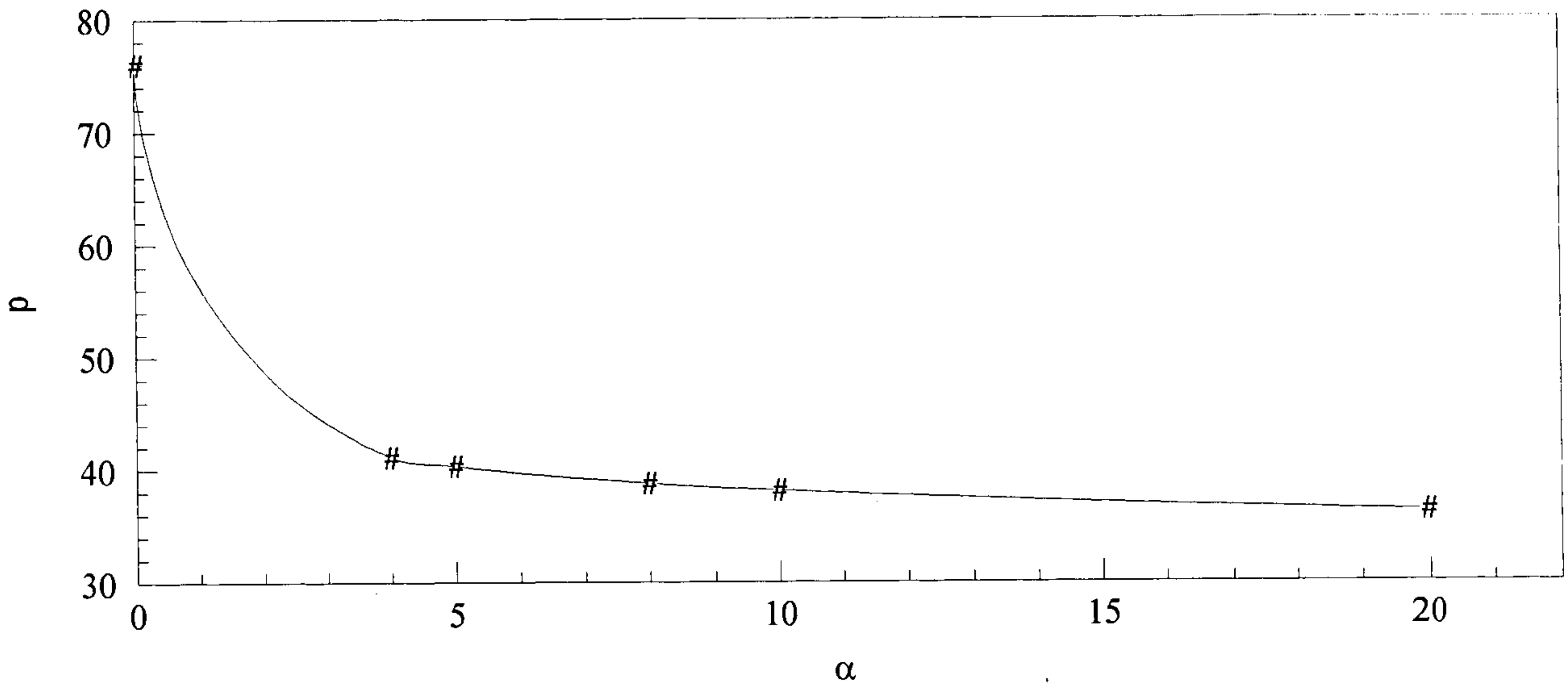


Figure (5.24) : Buckling load p versus ratio factor α for clamped-clamped column on elastic foundation with N=20, k1=600 , and starting C value = 0.2, using the power series method.

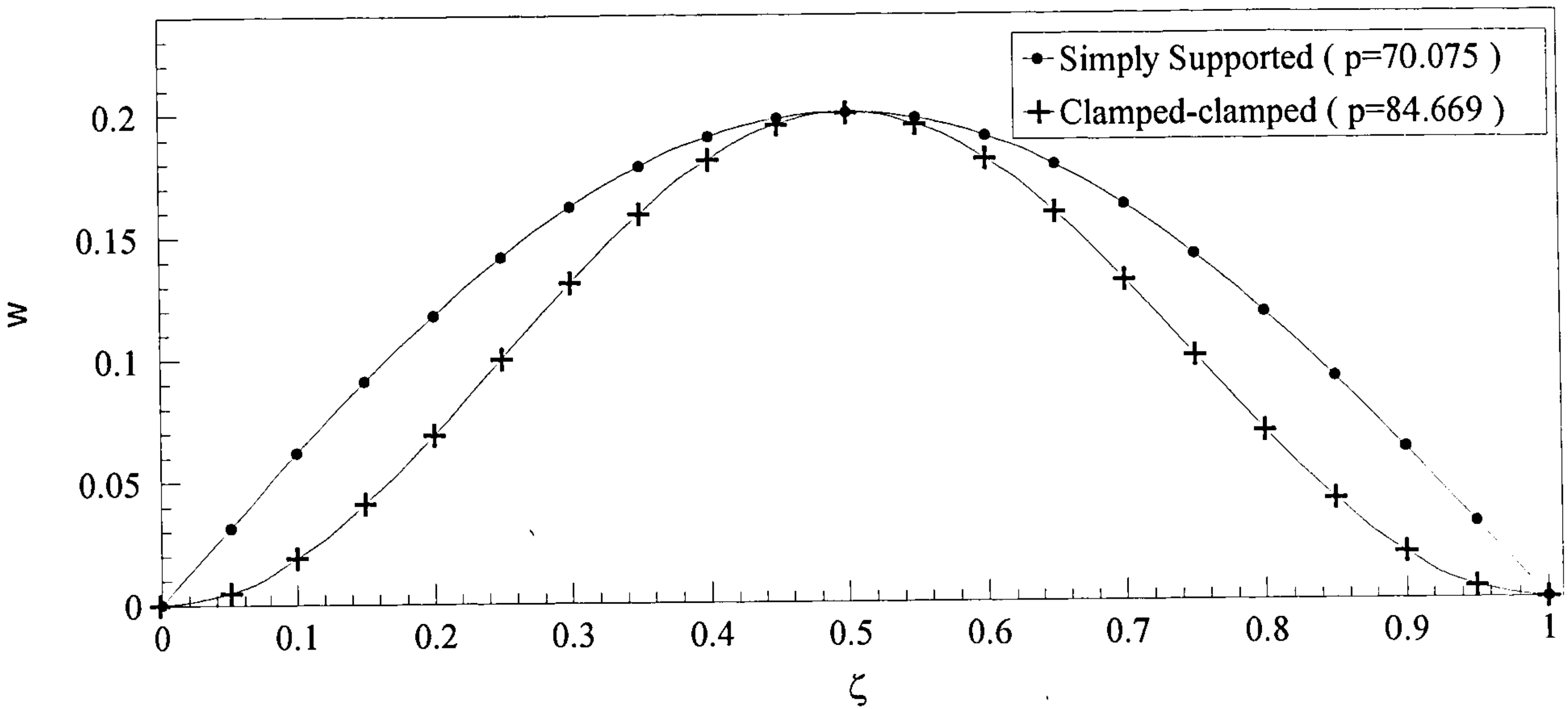


Figure (5.25) : Mode shapes of deflection for both simply supported and clamped-clamped column with k1=600, α=0.6 and midpoint deflection C=0.2, using the trial function method.

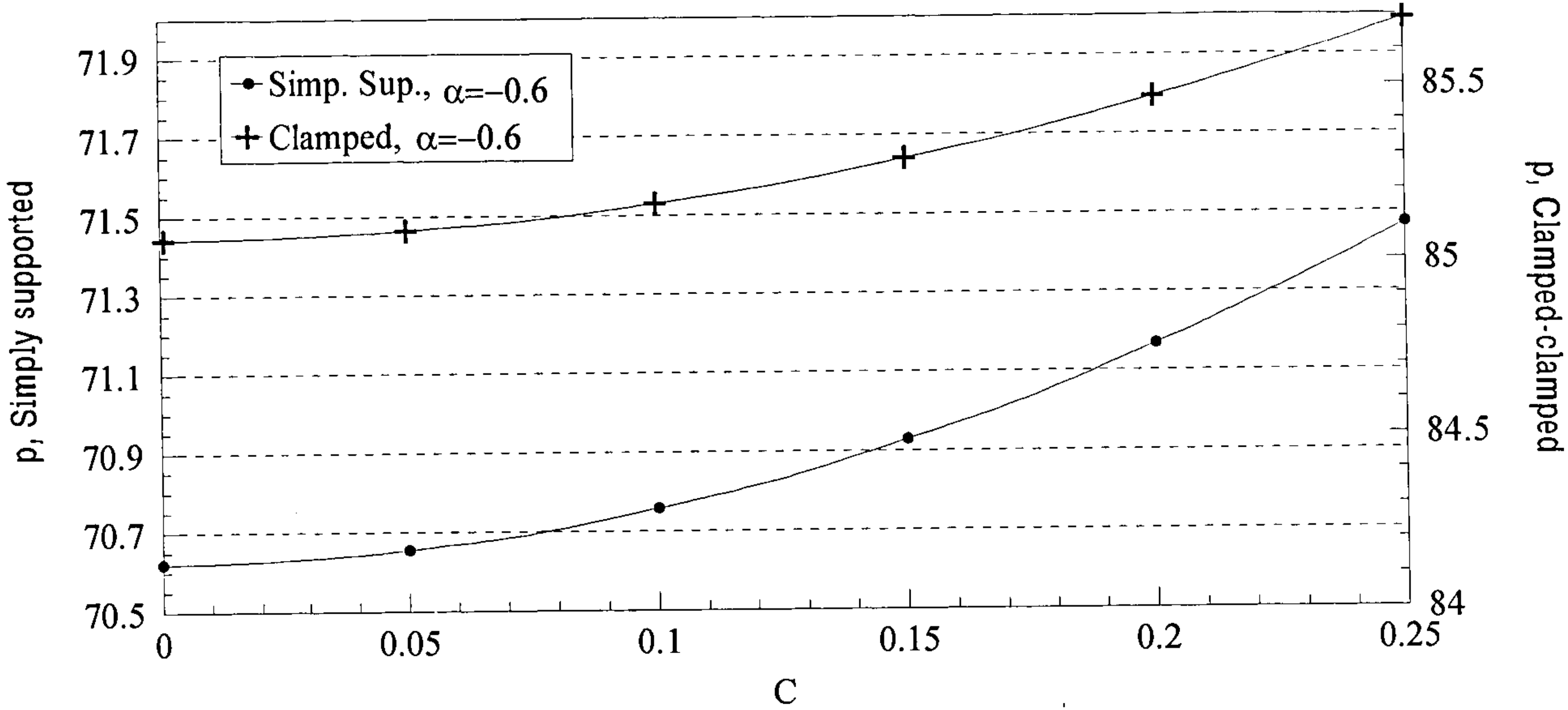


Figure (5.26) : Buckling load p versus mid point deflection C for both simply supported and clamped-clamped column with k1=600, α=-0.6, using the trial function method.

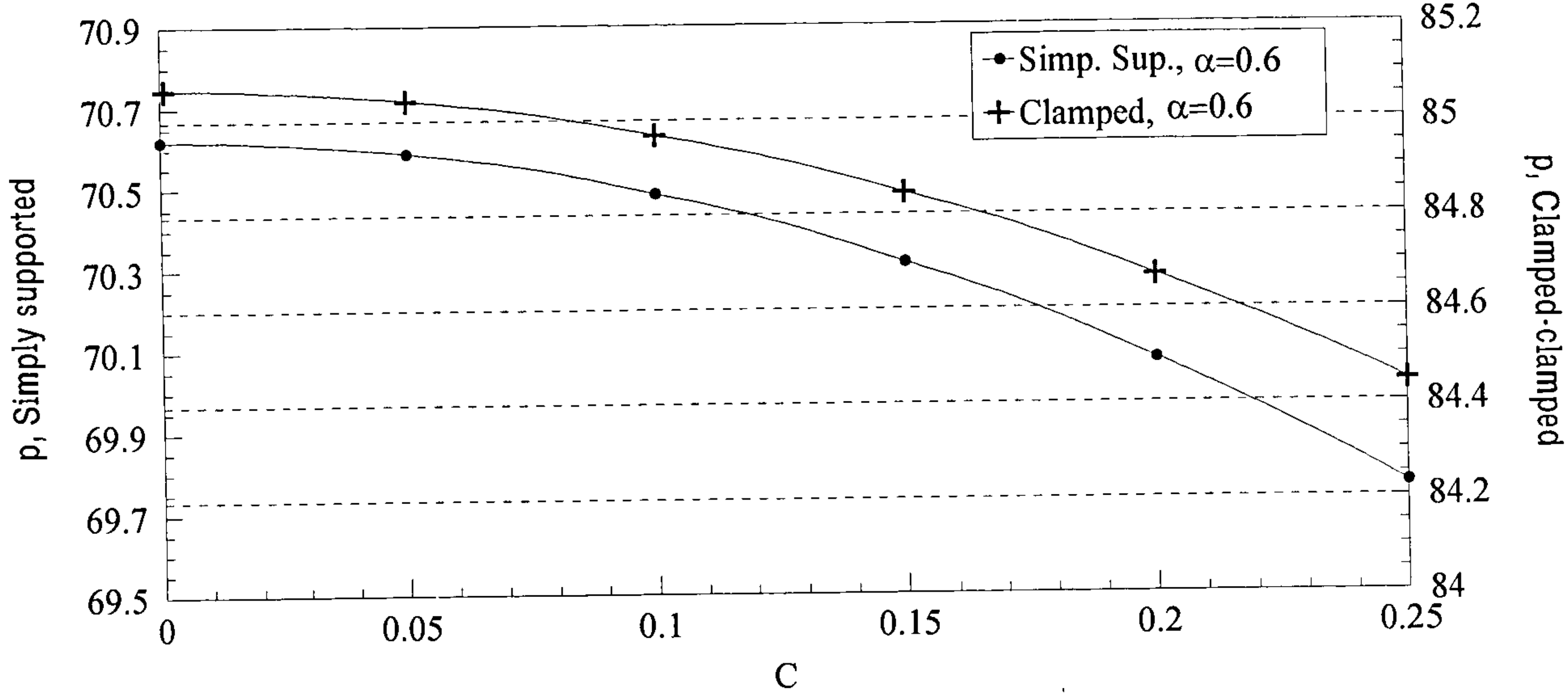


Figure (5.27) : Buckling load p versus midpoint deflection C for both simply supported and clamped-clamped column with k1=600, α=0.6, using the trial function method.

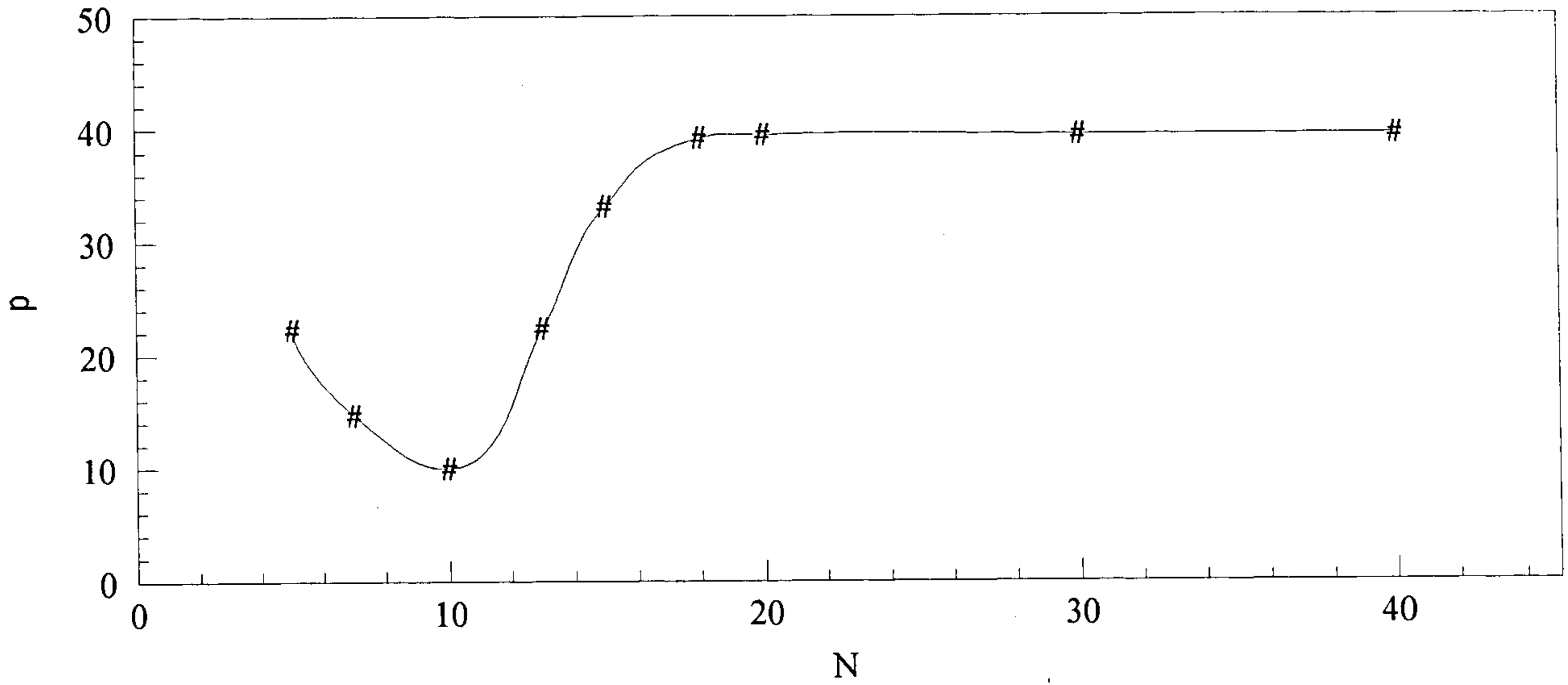


Figure (5.28) : Solution convergence showing the buckling load p versus the polynomial degree N for clamped-clamped column on elastic foundation for k1=0 , α =0 and starting C value = 0.3, using the power series method.

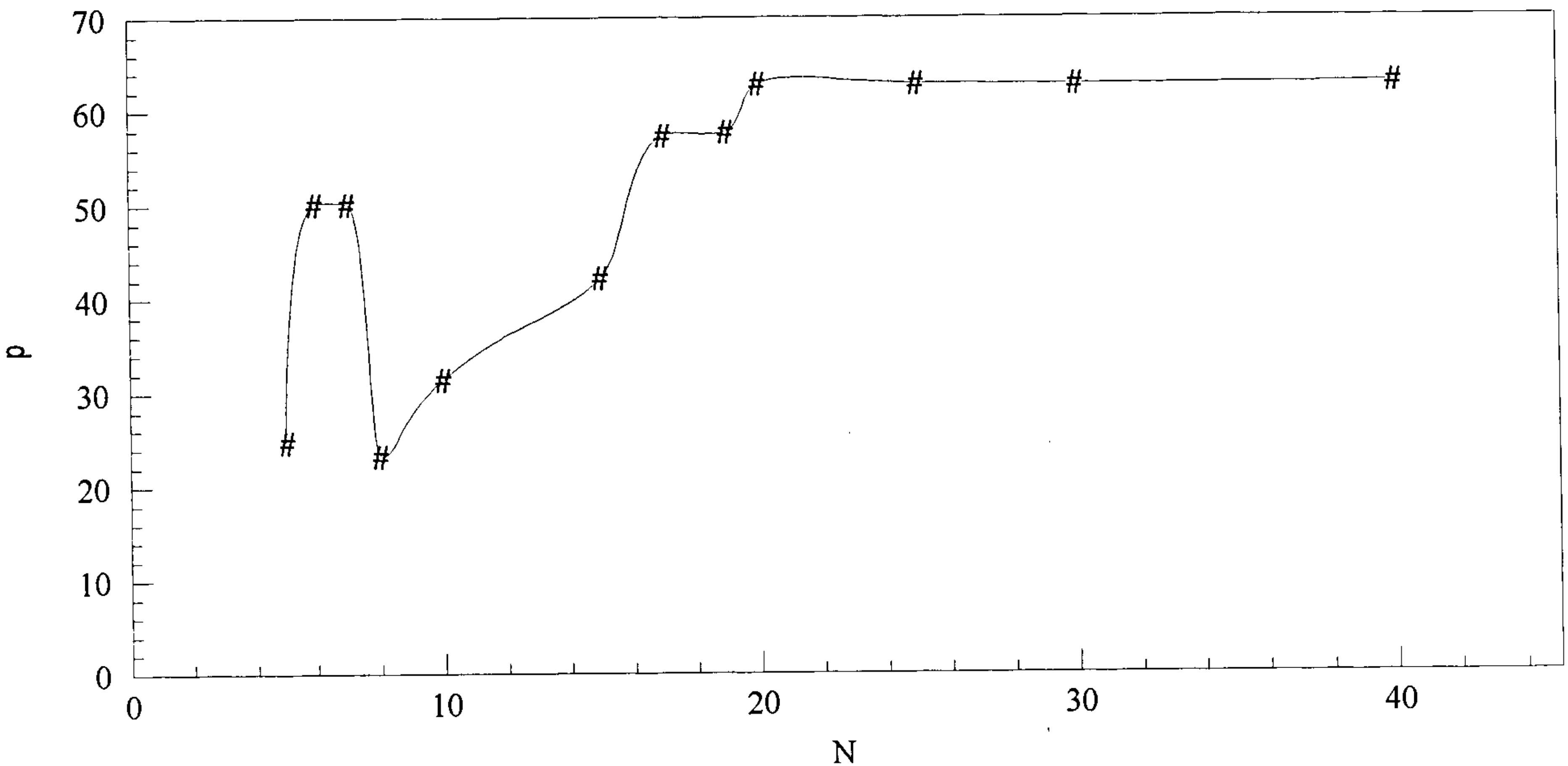


Figure (5.29) : Solution convergence showing the buckling load p versus the polynomial degree N for clamped-clamped column on elastic foundation for k1=200 , α =0 and starting C value = 0.3, using the power series method.

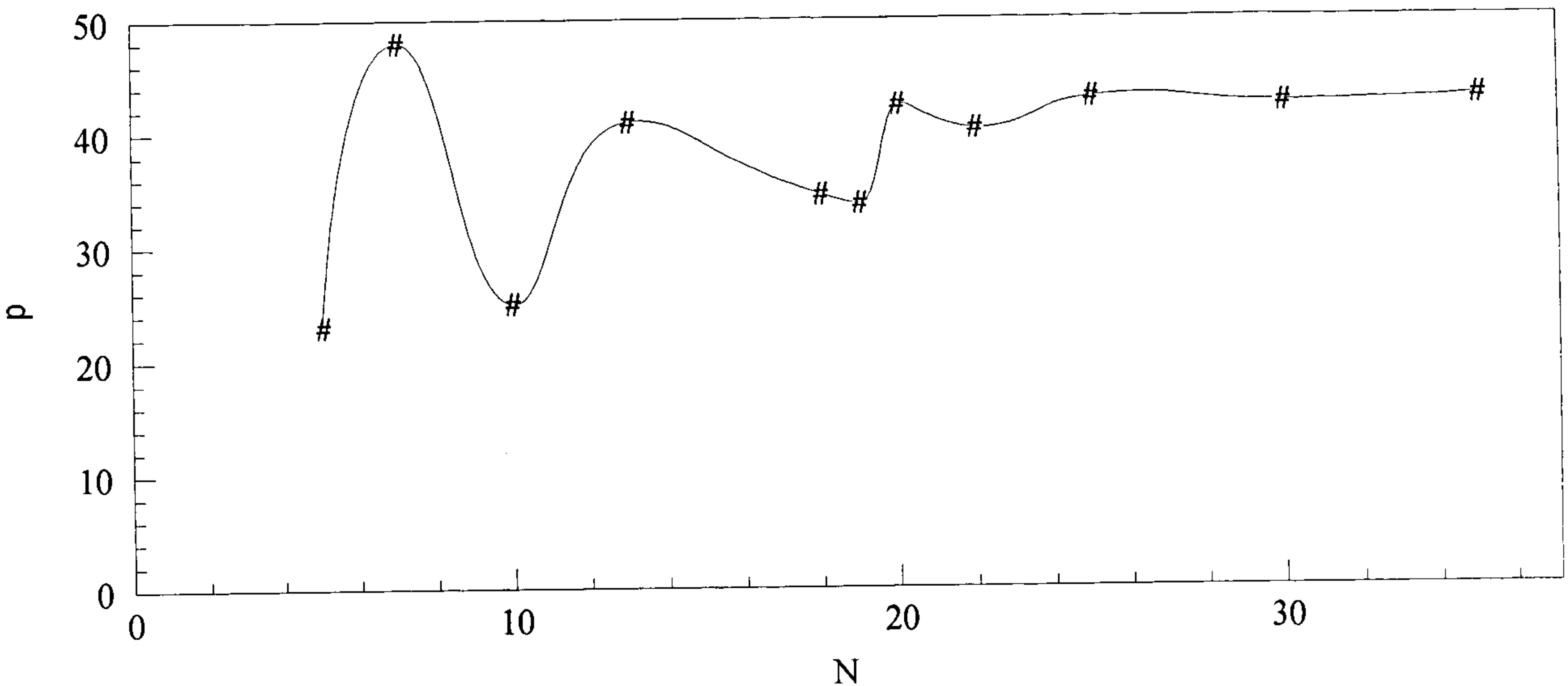


Figure (5.30) : Solution convergence showing the buckling load p versus the polynomial degree N for clamped-clamped column on elastic foundation for k1=100 , α =3 and starting C value = 0.2, using the power series method.

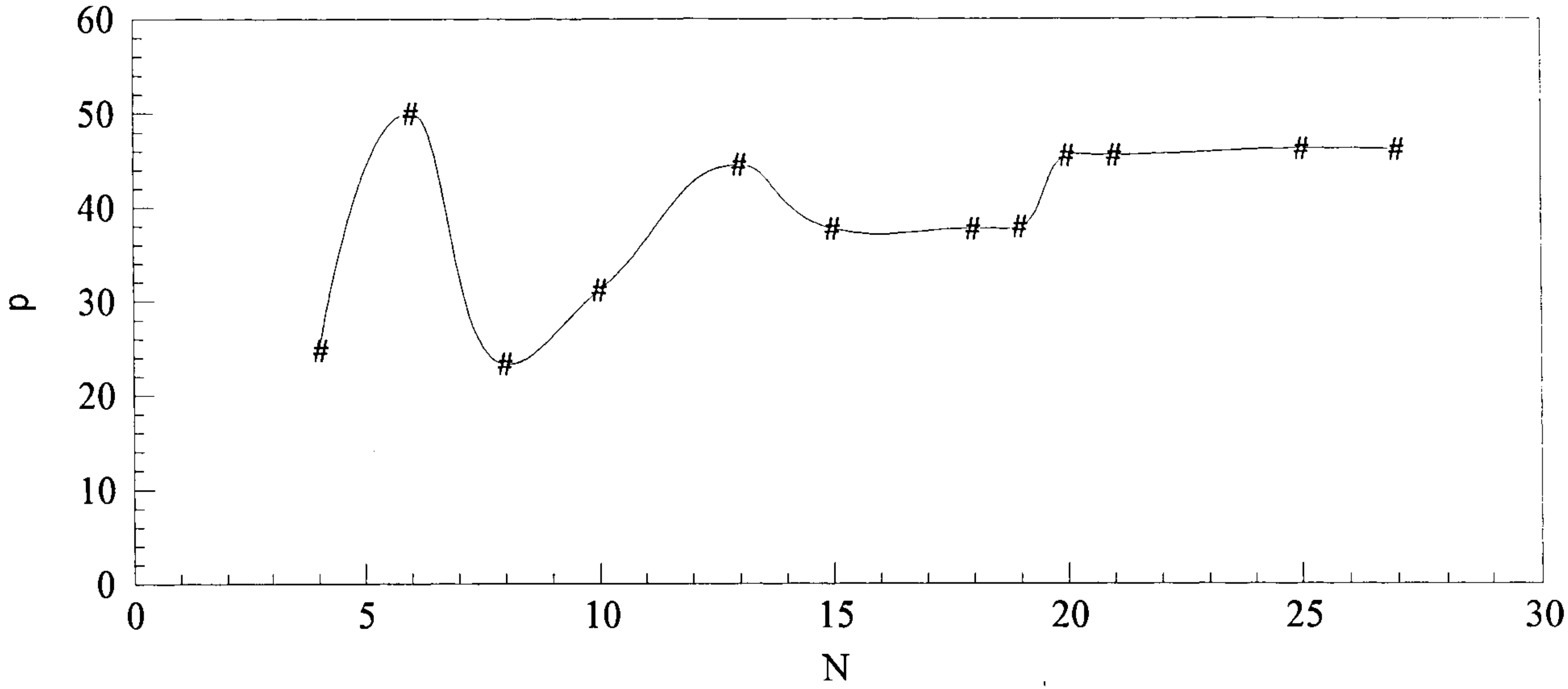


Figure (5.31) : Solution convergence showing the buckling load p versus the polynomial degree N for clamped-clamped column on elastic foundation for k1=200 , α =0.6 and starting C value = 0.3, using the power series method.

Chapter Six

CONCLUSIONS AND RECOMMENDATIONS

6.1 CONCLUSIONS.

Several important points have emerged from this work, which can be summarized as follows:

1- The power series method is a very powerful method. It can be computerized easily. Its results suffer from truncation errors and round off errors due to large mathematical operations since the problem is non-linear.

2- The trial function method is a good approximate method to find the buckling loads and the corresponding mode shapes. A closed-form formulas for the buckling loads is a good advantage of this method.

3- Increasing the linear foundation modulus (k_1) or decreasing the ratio factor (α) makes the structure to be more hardened, and hence, needs more axial compressive load to be buckled.

4- The overall foundation modulus (linear and nonlinear parts) has a great effect on the mode shapes. Increasing this modulus will decrease the wave length of the mode shape and transform it into another form.

5- The boundary conditions have great effect on determining the buckling load and the mode shape of the structure. It was found that the clamped-clamped column need more axial compressive load to be buckled than the simply supported columns. Furthermore, the level of deflection for clamped-clamped columns is less than the level for simply supported.

6- The running time of the power series program was mainly dependent on the polynomial order (N). It was found from the convergence study of the power series solution that the 20^{th} polynomial order is the most suitable one for establishing a solution to the problem, i.e. after which the non-dimensional buckling load will reach its steady state value.

7- The trend of solution of both trial function method and power series method is identical.

8- The trial function method is an approximate method, and gives the solution at integer mode shapes (for a specified values of k_1), while the power series method is an exact method which gives the accurate solution at any value of k_1 although it needs more running time if it was computerized.

6.2 RECOMMENDATIONS.

The present work suggests that the following points need to be investigated:

1- Take the non-linear terms of the functional ($w'^2 w''^2$ and w'^4) into consideration to study:

(a) The case of large deflection problems.

(b) The case when the column behaves as a very hardened or very softened structure, since the error becomes sensible if these terms are neglected.

2- Apply another forms of trial functions which satisfy the boundary conditions to the problem and compare their results.

3- Try to solve a similar problem but with initial imperfection types and other boundary conditions by using the power series method to study how much it is applicable to solve such kinds of problems.

4- Solve the buckling problem which is under consideration by using a finite-difference method. A detailed and complete formulation of this method is found in Appendix (B).

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Appendix A

PROGRAMS AND SAMPLE RUNS.

A.1 PROGRAM (1).

THIS PROGRAM FINDS THE MODE SHAPE NUMBER FOR DIFFERENT LINEAR FOUNDATION COEFFICIENTS AND RATIO FACTORS WITH THEIR CORRESPONDING BUCKLING LOADS FOR SIMPLY SUPPORTED COLUMNS ON ELASTIC FOUNDATION USING TRAIL FUNCTION METHOD DISCUSSED IN CHAPTER THREE.

```
      REAL MA
      OPEN(1,FILE='HEHE.OUT',STATUS='NEW')
      DO 3 K1=1400,1900,300
      WRITE(1,*)'~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
     '
      WRITE(1,*)'    **SIMPLY SUPPORTED**'
      WRITE(1,*)'     **** FOR  K1= ',K1
      WRITE(1,*)
      PI=22./7.
      DO 1 C=0.1,0.3,0.1
      WRITE(1,*)'   ALPHA                M       P       C
     '
      WRITE(1,*)'##############################################
     '
      DO 2 ALPHA=0.3,2.,0.4
```

```
      KEQ=K1*(1.-(3.*ALPHA*C**2)/8.)
      IF (KEQ.GT.0.0) MA=(KEQ**0.25)/PI
      IF (KEQ.GT.0.0) M=INT(MA)
      IF (KEQ.GT.0.0) P=(MA*PI)**2+KEQ/(MA*PI)**2
      IF (KEQ.GT.0.0) WRITE(1,*)ALPHA,M,P,C
    2 CONTINUE
    1 CONTINUE
    3 CONTINUE
      STOP
      END
```

SAMPLE RUN

```
**SIMPLY SUPPORTED**
 ****  FOR   K1=          1400

ALPHA                   M        P              C
####################################################
0.3000000               1    74.77968       0.1000000
0.7000000               1    74.72617       0.1000000
 1.100000               1    74.67262       0.1000000
 1.500000               1    74.61903       0.1000000
 1.900000               1    74.56541       0.1000000
ALPHA                   M        P              C
####################################################
0.3000000               1    74.64584       0.2000000
0.7000000               1    74.43118       0.2000000
 1.100000               1    74.18895       0.2000000
 1.500000               1    73.97297       0.2000000
 1.900000               1    73.75636       0.2000000
ALPHA                   M        P              C
####################################################
0.3000000               1    74.43118       0.3000000
0.7000000               1    73.91888       0.3000000
 1.100000               1    73.43024       0.3000000
 1.500000               1    72.91090       0.3000000
 1.900000               1    72.38785       0.3000000
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~


**SIMPLY SUPPORTED**
 ****  FOR   K1=          1700

ALPHA                   M        P              C
####################################################
0.3000000               2    82.41359       0.1000000
0.7000000               2    82.34076       0.1000000
 1.100000               2    82.26786       0.1000000
```

1.500000	2	82.21922	0.1000000
1.900000	2	82.14621	0.1000000
ALPHA	M	P	C

##

0.3000000	2	82.26786	0.2000000
0.7000000	2	82.02439	0.2000000
1.100000	2	81.75574	0.2000000
1.500000	2	81.51073	0.2000000
1.900000	2	81.26500	0.2000000
ALPHA	M	P	C

##

0.3000000	2	82.02439	0.3000000
0.7000000	2	81.46165	0.3000000
1.100000	2	80.89500	0.3000000
1.500000	2	80.32434	0.3000000
1.900000	2	79.74961	0.3000000

A.2 PROGRAM (2).

THIS PROGRAM FINDS THE BUCKLING LOADS BASED ON THE DIFFERENT FUNCTIONALS DISCUSSED IN THE THESIS USING TRAIL FUNCTIONS FOR DIFFERENT VALUES OF LINEAR FOUNDATION COEFFICIENTS, RATIO FACTORS AND MID-POINT DEFLECTIONS, FOR SIMPLY SUPPORTED COLUMNS ON ELASTIC FOUNDATION.

```
      OPEN(1,FILE='HOHO.OUT',STATUS='NEW')
      READ(*,*) M
      DO 3 K1=0,600,200
      WRITE(1,*)
      WRITE(1,*)'     **SIMPLY SUPPORTED**'
      WRITE(1,*)'      **** FOR  K1= ',K1
      WRITE(1,*)
      PI=22./7.
      DO 1 ALPHA=-1.,1.,0.2
      WRITE(1,*)
      WRITE(1,*)'     ALPHA     M      C            P1
     +         P2          P3    % ERROR'
      WRITE(1,*)'############################################
     +###########################'
      DO 2 C=0,0.25,0.05
      K3=ALPHA*K1
      P1=((M*PI)**4+K1)/((M*PI)**2)
    7 IF(C.NE.0) THEN
      P2=(16.*(M*PI)**4+16.*K1+(4.*(M*PI)**6-6.*K3)*C**2)/
     +(16.*(M*PI)**2+3.*((M*PI)**4)*C**2)
      ELSE
```

```
      P2=P1
      ENDIF
      P3=(8.*(M*PI)**4+8.*K1-3.*K3*C**2)/(8.*(M*PI)**2)
      ERROR=((ABS(P3-P2))/P3)*100.
      WRITE(1,4)ALPHA,M,C,P1,P2,P3,ERROR
    4
FORMAT(5X,F5.2,4X,I1,4X,F4.2,2X,3(3X,F7.3),2X,F5.2,'%')
    2 CONTINUE
    1 CONTINUE
    3 CONTINUE
      STOP
      END
```

SAMPLE RUN

SIMPLY SUPPORTED

**** FOR K1= 600

ALPHA	M	C	P1	P2	P3	% ERROR
-0.60	1	0.00	70.621	70.621	70.621	0.00%
-0.60	1	0.05	70.621	70.391	70.656	0.37%
-0.60	1	0.10	70.621	69.711	70.758	1.48%
-0.60	1	0.15	70.621	68.618	70.929	3.26%
-0.60	1	0.20	70.621	67.168	71.168	5.62%
-0.60	1	0.25	70.621	65.427	71.476	8.46%

ALPHA	M	C	P1	P2	P3	% ERROR
-0.30	1	0.00	70.621	70.621	70.621	0.00%
-0.30	1	0.05	70.621	70.374	70.638	0.37%
-0.30	1	0.10	70.621	69.644	70.690	1.48%
-0.30	1	0.15	70.621	68.471	70.775	3.26%
-0.30	1	0.20	70.621	66.913	70.895	5.62%
-0.30	1	0.25	70.621	65.044	71.048	8.45%

ALPHA	M	C	P1	P2	P3	% ERROR
0.00	1	0.00	70.621	70.621	70.621	0.00%
0.00	1	0.05	70.621	70.357	70.621	0.37%
0.00	1	0.10	70.621	69.577	70.621	1.48%
0.00	1	0.15	70.621	68.323	70.621	3.25%
0.00	1	0.20	70.621	66.659	70.621	5.61%
0.00	1	0.25	70.621	64.661	70.621	8.44%

ALPHA	M	C	P1	P2	P3	% ERROR
0.30	1	0.00	70.621	70.621	70.621	0.00%
0.30	1	0.05	70.621	70.340	70.604	0.37%
0.30	1	0.10	70.621	69.510	70.553	1.48%
0.30	1	0.15	70.621	68.175	70.468	3.25%
0.30	1	0.20	70.621	66.404	70.348	5.61%
0.30	1	0.25	70.621	64.278	70.194	8.43%

ALPHA	M	C	P1	P2	P3	% ERROR
0.60	1	0.00	70.621	70.621	70.621	0.00%
0.60	1	0.05	70.621	70.323	70.587	0.37%
0.60	1	0.10	70.621	69.442	70.485	1.48%
0.60	1	0.15	70.621	68.028	70.314	3.25%
0.60	1	0.20	70.621	66.150	70.075	5.60%
0.60	1	0.25	70.621	63.896	69.767	8.42%

A.3 PROGRAM (3)

THIS PROGRAM FINDS THE BUCKLING LOADS BASED ON THE DIFFERENT FUNCTIONALS DISCUSSED IN THE THESIS USING TRAIL FUNCTIONS FOR DIFFERENT VALUES OF LINEAR FOUNDATION COEFFICIENTS,RATIO FACTORS AND MID-POINT DEFLECTIONS, FOR CLAMPED-CLAMPED COLUMNS ON ELASTIC FOUNDATION

```
      OPEN(1,FILE='HOHOHO.OUT',STATUS='NEW')
      READ(*,*) M
      DO 3 K1=0,800,200
      WRITE(1,*)
      WRITE(1,*)'     **CLAMPED-CLAMPED**'
      WRITE(1,*)'      **** FOR  K1= ',K1
      WRITE(1,*)
      PI=22./7.
      DO 1 ALPHA=-1.,1.,0.2
      WRITE(1,*)
      WRITE(1,*)'     ALPHA     M      C           P1
     +          P2          P3    % ERROR'
      WRITE(1,*)'###########################################
#
     +#####################'
      DO 2 C=0,0.25,0.05
      K3=ALPHA*K1
      P1=(16.*(M*PI)**4+3.*K1)/(4.*(M*PI)**2)
    7 IF(C.NE.0) THEN
      P2=(512.*(M*PI)**4+96.*K1+(128.*(M*PI)**6-
     +35.*K3)*C**2)/(128.*(M*PI)**2+32.*((M*PI)**4)*C**2)
```

```
      ELSE
      P2=P1
      ENDIF
      P3=(512.*(M*PI)**4+96.*K1-
35.*K3*C**2)/(128.*(M*PI)**2)
      ERROR=((ABS(P3-P2))/P3)*100.
      WRITE(1,4)ALPHA,M,C,P1,P2,P3,ERROR
   4
FORMAT(5X,F5.2,4X,I1,4X,F4.2,2X,3(3X,F7.3),2X,F5.2,'%')
   2 CONTINUE
   1 CONTINUE
   3 CONTINUE
      STOP
      END
```

A.4 PROGRAM (4)

THIS PROGRAM IS DESIGNED TO FIND THE NON-LINEAR BUCKLING LOAD NORMALIZED EIGEN-VECTOR OF DEFLECTION AND THE SLOPE VECTOR OF A CLAMPED-CLAMPED COLUMN ON ELASTIC FOUNDATION USING THE POWER SERIES METHOD.

```
      DOUBLE PRECISION C(0:100),V(0:150),D(0:100),R(0:100),
     +A(0:100),Q(0:100),F(0:200),U(0:20),UP(0:20),
     +NU(0:20),MAX,SUM,ERROR
      REAL inc,K1,K3
      OPEN(1,FILE='CLAMP13.OUT',STATUS='NEW')
      READ(*,*)C2,C3, K1,ALPHA,N,TOL,start,inc
      K3=K1*ALPHA
      C(0)=0.0
      C(1)=0.0
      C(2)=C2
      C(3)=C3
      DO 17 P=start,100,inc

C     ********** [ NONLINEAR TERM V(I) ] *********

      DO 1 I=0,N-4
      SUM=0.0
      DO 2 J=0,I
      DO 3 K=0,J
      SUM=SUM+C(K)*C(I-J)*C(J-K)
    3 CONTINUE
    2 CONTINUE
      V(I)=SUM

C     ********* [ RECURRENCE FORMULA ] ***********
```

```
      C(I+4)=(K3*V(I)-K1*C(I)-P*(I+2)*(I+1)*C(I+2))/
     +((I+4)*(I+3)*(I+2)*(I+1))
  1   CONTINUE

      DO 4 I=N-3,3*N
      SUM=0.0
      DO 5 J=0,I
      DO 6 K=0,J
      SUM=SUM+C(K)*C(I-J)*C(J-K)
  6   CONTINUE
  5   CONTINUE
      V(I)=SUM
  4   CONTINUE

C     ******* [TERMS OF BUCKLING-LOAD FORMULA ]*****

      SUM1=0.0
      DO 616 I=0,2*(N-1)
      SUM=0.0
      DO 617 J=0,I
      SUM=SUM+(J+1)*(I+1-J)*C(J+1)*C(I+1-J)
 617  CONTINUE
      D(I)=SUM
      SUM1=SUM1+D(I)/(I+1)
 616  CONTINUE

C     ------------------------------------------------------------

      SUM2=0.0
      DO 618 I=0,2*(N-2)
      SUM=0.0
      DO 619 J=0,I
      SUM=SUM+(J+1)*(J+2)*(I+1-J)*(I+2-J)*C(J+2)*C(I+2-J)
 619  CONTINUE
      R(I)=SUM
      SUM2=SUM2+R(I)/(I+1)
```

```
  618 CONTINUE
C      ------------------------------------------------------------
       SUM3=0.0
       DO 620 I=0,2*N
       SUM=0.0
       DO 621 J=0,I
       SUM=SUM+C(J)*C(I-J)
  621  CONTINUE
       Q(I)=SUM
       SUM3=SUM3+Q(I)/(I+1)
  620  CONTINUE
C      ------------------------------------------------------------
       SUM4=0.0
       DO 622 I=0,4*N
       SUM=0.0
       DO 623 J=0,I
       DO 624 K=0,J
       DO 625 M=0,K
       SUM=SUM+C(M)*C(K-M)*C(J-K)*C(I-J)
  625  CONTINUE
  624  CONTINUE
  623  CONTINUE
       F(I)=SUM
       SUM4=SUM4+F(I)/(I+1)
  622  CONTINUE
C      ------------------------------------------------------------
       ERROR=P*SUM1-(SUM2+K1*SUM3-0.5*K3*SUM4)
       WRITE(*,*)P,ERROR
       IF(ABS(ERROR).LE.TOL)THEN
       GO TO 18
       ELSE
```

```
      ENDIF
   17 CONTINUE

C     ****** [ CONVERTING C-COEFF's INTO A-COEFF's ]******

   18 A(2)=C(2)
      A(3)=C(3)+2.*A(2)
      DO 188 I=4,N-4
      A(I)=C(I)-A(I-2)+2.*A(I-1)
  188 CONTINUE
      A(N-3)=C(N-1)+2.*C(N)
      A(N-2)=C(N)
      A(N-1)=0.0
      A(N)=0.0

C     ------------------------------------------------------------

      WRITE(1,*)'_____________________________________________
     +___________________'
      WRITE(1,*)'**** RESULTS OF THE CLAMPED-CLAMPED COLUMN
     +PROGRAM ****'
      WRITE(1,*)
      WRITE(1,*)'_____________________________________________
     +___________________'
      WRITE(1,66)N,K1,ALPHA
      WRITE(1,*)'** P=',P,'           ** ERROR=',ERROR
      WRITE(1,*)
      WRITE(1,*)'_____________________________________________
     +___________________'
      WRITE(1,*)'** THE COEFFICIENTS C0,C1,C2,C3,......,etc
     +ARE:'
      WRITE(1,*)
      WRITE(1,*)(C(I),I=0,N)
      WRITE(1,*)
      WRITE(1,*)'_____________________________________________
     +___________________'
```

```
      WRITE(1,*)'** THE COEFFICIENTS A0,A1,A2,A3,......,etc
     +ARE:'
      WRITE(1,*)
      WRITE(1,*)(A(I),I=0,N)

C     *** [ FINDING THE DEFLECTION AND SLOPE VECTORS ] ****

      DO 88 ZETA=0,1,0.1
      SUM=0.0
      SUMP=0.0
      DO 77 K=2,N
      SUM=SUM+A(K)*ZETA**K
      SUMP=SUMP+k*A(K)*ZETA**(K-1)
   77 CONTINUE
      U(10*ZETA)=SUM*(ZETA-1.)**2
      UP(10*ZETA)=SUMP*(ZETA-1.)**2+2.*SUM*(ZETA-1.)
   88 CONTINUE

C     ****** [ NORMALIZING THE DEFLECTION VECTOR ] *******

      MAX=0.0
      DO 3800 ZETA=0,1,0.1
      IF(ABS(MAX).LT.ABS(U(10*ZETA))) MAX=U(10*ZETA)
 3800 CONTINUE
      DO 3900 ZETA=0,1,0.1
      NU(10*ZETA)=U(10*ZETA)/MAX
 3900 CONTINUE

C     -------------------------------------------------------

      WRITE(1,*)
      WRITE(1,*)'____________________________________________________
     +__________________'
      WRITE(1,*)'** THE DEFLECTION VECTOR IS:'
      WRITE(1,*)(U(I),I=0,10)
      WRITE(1,*)
```

```
      WRITE(1,*)'_____________________________________________
     +___________________'
      WRITE(1,*)'** THE SLOPE VECTOR IS:'
      WRITE(1,*)(UP(I),I=0,10)
      WRITE(1,*)
      WRITE(1,*)'_____________________________________________
     +___________________'
      WRITE(1,*)'** THE NORMALIZED DEFLECTION VECTOR IS:'
      WRITE(1,*)(NU(I),I=0,10)
      WRITE(1,*)'~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
     +~~~~~~~~~~~~~~~~~~~'

C     ----------------------------------------------------------------

   66 FORMAT(1X,'** POLYNOMIAL DEGREE=',I2,8X,'** K1=',F5.1,
     +7X,'** ALPHA=',F4.1)
      STOP
      END
```

SAMPLE RUN (1)

```
** POLYNOMIAL DEGREE=20          ** K1=   0.0        ** ALPHA= 0.0
** P=    39.50000                ** ERROR= -3.0621886253356934E-06
```

```
** THE COEFFICIENTS C0,C1,C2,C3,......,etc ARE:

 0.0000000000000000E+00  0.0000000000000000E+00  0.2000000029802322
 0.0000000000000000E+00 -0.6583333431432645      0.000000000000E+00
 0.8668055684719649      0.0000000000000000E+00 -0.6114074991900467
 0.0000000000000000E+00  0.2683399579778538      0.000000000000E+00
-8.0298699546403222E-02  0.0000000000000000E+00  1.742746501144E-02
 0.0000000000000000E+00 -2.8682702831335995E-03  0.000000000000E+00
 3.7025057576397771E-04  0.0000000000000000E+00 -3.848657300704E-05
```

```
** THE COEFFICIENTS A0,A1,A2,A3,......,etc ARE:

 0.0000000000000000E+00  0.0000000000000000E+00  0.200000002980232
 0.4000000059604645     -5.8333334202567741E-02 -0.516666674365600
-0.1081944460566673      0.3002777822522653      9.73425113711151E-02
-0.1055927595099627     -4.0188072413222857E-02  2.5216614683516E-02
 1.0322602233853534E-02 -4.5714102158098818E-03 -2.0379576540286E-03
 4.9549490775259702E-04  1.6067718640023695E-04 -7.6973146014090E-05
-3.8486573007045052E-05  0.0000000000000000E+00  0.0000000000000E+00
```

```
** THE DEFLECTION VECTOR IS:
 0.0000000000000000E+00  1.9350274058541683E-03  7.0006052061411E-03
 1.3260829942823044E-02  1.8323247974497244E-02  2.0253161507508E-02
 1.8313028787481136E-02  1.3244358589498466E-02  6.9845341263439E-03
 1.9265848357003849E-03  2.8153916605800178E-15
```

```
** THE SLOPE VECTOR IS:
 0.0000000000000000E+00  3.7418187186149918E-02  6.0536345971958E-02
 6.0519457722971485E-02  3.7373979570527325E-02 -5.4644714439652E-05
-3.7462210431771153E-02 -6.0551692006742019E-02 -6.0494802774432E-02
-3.7313569773251916E-02  4.7234433785805815E-08
```

```
** THE NORMALIZED DEFLECTION VECTOR IS:
 0.0000000000000000E+00  9.5541992549499310E-02  0.3456549340973
 0.6547535770110227      0.9047105049601450      1.0000000000000
 0.9042059325252528      0.6539403038181697      0.3448614244129
 9.5125140585392197E-02  1.3900998417142347E-13
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
```

SAMPLE RUN (2)

```
** POLYNOMIAL DEGREE=20          ** K1=100.0          ** ALPHA= 0.0
** P=    46.90000                ** ERROR=  7.4696540832519531E-04
```

```
** THE COEFFICIENTS C0,C1,C2,C3,......,etc ARE:

 0.0000000000000000E+00  0.0000000000000000E+00  0.2000000029802
 0.0000000000000000E+00 -0.7816667037457232      0.0000000000000E+00
 1.166450163159415       0.0000000000000000E+00 -0.9303743164164
 0.0000000000000000E+00  0.4616845683741192      0.0000000000000E+00
-0.1562065115233526      0.0000000000000000E+00  3.8331457573164E-02
 0.0000000000000000E+00 -7.1329907130753398E-03  0.0000000000000E+00
 1.0410647836200927E-03  0.0000000000000000E+00 -1.2235499201069E-04
```

```
** THE COEFFICIENTS A0,A1,A2,A3,......,etc ARE:

 0.0000000000000000E+00  0.0000000000000000E+00  0.2000000029802
 0.4000000059604645     -0.1816666948050264     -0.7633333955705
-0.1785499331765936      0.4062335292173302      6.0642675194789E-02
-0.2849481788277509     -0.1688544644761723     -5.2760750124593E-02
-9.2873547296367581E-02 -0.1329863444681415     -0.1347676840667
-0.1365490236653611     -0.1454633539770462     -2.4470998402139E-04
-1.2235499201069655E-04  0.0000000000000000E+00  0.0000000000000E+00
```

```
** THE DEFLECTION VECTOR IS:
 0.0000000000000000E+00  1.9229905433035677E-03  6.8216519829970E-03
 1.2460445690131044E-02  1.6203290606864402E-02  1.6152921419633E-02
 1.1982156581024819E-02  5.1570909175950448E-03 -1.3045141785732E-03
-3.2263564277788736E-03 -1.7141786177496979E-14
```

```
** THE SLOPE VECTOR IS:
 0.0000000000000000E+00  3.6942580374029100E-02  5.7133313655812E-02
 5.1046698198064054E-02  2.0501237328423032E-02 -2.2088256880814E-02
-5.8826428752824064E-02 -7.2608783480062344E-02 -4.9387925518022E-02
 1.6345591507850614E-02 -2.8759165725017909E-07
```

** THE NORMALIZED DEFLECTION VECTOR IS:

0.0000000000000000E+00	0.1186790134152693	0.4210041125909
0.7690071104971894	1.0000000000000000	0.9968914223380
0.7394890872319891	0.3182742964204001	-8.0509213234785E-02
-0.1991173586933047	-1.0579200604002616E-12	

~~~~~~
~~~~~~

SAMPLE RUN (3)

```
** POLYNOMIAL DEGREE=20          ** K1=100.0          ** ALPHA= 3.0
** P=    42.40000                ** ERROR= -5.8412551879882813E-06
```

```
** THE COEFFICIENTS C0,C1,C2,C3,......,etc ARE:

 0.0000000000000000E+00  0.0000000000000000E+00  0.2000000029802
 0.0000000000000000E+00 -0.7066667026281361      0.0000000000000E+00
 0.9432000859406283      0.0000000000000000E+00 -0.6720737137224
 0.0000000000000000E+00  0.2983833166317535      0.0000000000000E+00
-9.2328572373299890E-02  0.0000000000000000E+00  2.5422471854767E-02
 0.0000000000000000E+00 -1.2750919371236902E-02  0.0000000000000E+00
 1.2159219997440094E-02  0.0000000000000000E+00 -1.1453210399130E-02
```

```
** THE COEFFICIENTS A0,A1,A2,A3,......,etc ARE:

 0.0000000000000000E+00  0.0000000000000000E+00  0.2000000029802
 0.4000000059604645     -0.1066666936874393     -0.6133333933353
-0.1768000070426187      0.2597333792501058      2.4193051820410E-02
-0.2113472756092853     -0.1485042864072273     -8.5661297205169E-02
-0.1151468803764113     -0.1446324635476532     -0.1486955748641
-0.1527586861806020     -0.1695727168683133     -2.2906420798260E-02
-1.1453210399130084E-02  0.0000000000000000E+00  0.0000000000000E+00
```

```
** THE DEFLECTION VECTOR IS:
 0.0000000000000000E+00  1.9302698604828389E-03  6.9280087882440E-03
 1.2921212534338837E-02  1.7362051817061585E-02  1.8216471668920E-02
 1.4766758019624130E-02  7.9838310665892808E-03  6.1480120050078E-04
-2.7012224465773034E-03 -1.7387753805681672E-14
```

```
** THE SLOPE VECTOR IS:
 0.0000000000000000E+00  3.7229393148761315E-02  5.9130304545870E-02
 5.6312839228004284E-02  2.8973203194531859E-02 -1.3140516417588E-02
-5.4233089911581133E-02 -7.6896513299556863E-02 -6.2578629436489E-02
 4.3795298420860381E-03 -2.9171833179246131E-07
```

```
** THE NORMALIZED DEFLECTION VECTOR IS:
 0.0000000000000000E+00  0.1059628832391363     0.3803156239121
 0.7093147767129962      0.9530963038623728     1.0000000000000
 0.8106266838060715      0.4382753812973969     3.3749740985775E-02
-0.1482846127214603     -9.5450722410462462E-13
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
```

SAMPLE RUN (4)

```
** POLYNOMIAL DEGREE=20          ** K1=200.0        ** ALPHA=-0.6
** P=    41.20000                ** ERROR=  5.8402121067047119E-04
```

```
** THE COEFFICIENTS C0,C1,C2,C3,......,etc ARE:

 0.0000000000000000E+00  0.0000000000000000E+00  0.2000000029802
 0.0000000000000000E+00 -0.6866666896144551      0.0000000000000E+00
 0.8319111875379545      0.0000000000000000E+00 -0.5303029710867
 0.0000000000000000E+00  0.2095580855924653      0.0000000000000E+00
-5.5647540901155131E-02  0.0000000000000000E+00  8.9407952007620E-03
 0.0000000000000000E+00  1.6674936622284606E-03  0.0000000000000E+00
-3.6052802260900645E-03  0.0000000000000000E+00  3.3648624101003E-03
```

```
** THE COEFFICIENTS A0,A1,A2,A3,......,etc ARE:

 0.0000000000000000E+00  0.0000000000000000E+00  0.2000000029802
 0.4000000059604645     -8.6666680673758378E-02 -0.5733333673079
-0.2280888664042496      0.1171556344994821     -6.7902835683500E-02
-0.2529613058664824     -0.2284616904569994     -0.2039620750475
-0.2351100005391884     -0.2662579260308604     -0.2884650563217
-0.3106721866126805     -0.3312118232413620      6.7297248202006E-03
 3.3648624101003009E-03  0.0000000000000000E+00  0.0000000000000E+00
```

```
** THE DEFLECTION VECTOR IS:
 0.0000000000000000E+00  1.9321599818479464E-03  6.9532403096409E-03
 1.3010879045564573E-02  1.7502406564297432E-02  1.8203517080682E-02
 1.4094193304539724E-02  5.9302106274875158E-03 -3.0683700779540E-03
-6.0167164748324953E-03 -3.3336492520599666E-14
```

```
** THE SLOPE VECTOR IS:
 0.0000000000000000E+00  3.7302820478114977E-02  5.9570706180320E-02
 5.7081558507191490E-02  2.8898953308853326E-02 -1.6664719706127E-02
-6.4348797650433920E-02 -9.3835218618751871E-02 -7.4500904162824E-02
 2.9370355213159881E-02 -5.5929395656312991E-07
```

** THE NORMALIZED DEFLECTION VECTOR IS:

```
 0.0000000000000000E+00   0.1061421248039149      0.3819723561563
 0.7147453422268611       0.9614848870535788      1.0000000000000
 0.7742566033844529       0.3257727944112950     -0.1685591891036
-0.3305249446118066      -1.8313215173114608E-12
```

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Appendix B

FINITE DIFFERENCE METHOD.

B.1 INTRODUCTION

The problem of non-linear buckling on elastic foundation which is under consideration will only be formulated here by using a finite difference method.

The finite difference method is a numerical technique for solving differential equations. It essentially reduces a problem having infinite degrees of freedom to one with finite degrees of freedom, Gerald, *et. al* (1983). This is achieved by substituting an algebraic expression for each unknown function and its derivative.

Since each derivative is replaced by the value of the function at the reference and neighboring points, the accuracy of the derivatives, and hence of the solution, can be increased if these points are so chosen that they are as close to each other as possible. This technique has the disadvantage that it gives the value of the function only at discrete points. Thus, if an analytical expression has to be obtained for a deflected shape, a curve has then to be fitted passing through all these points.

B.2 DERIVATION OF FINITE ELEMENT RELATION USING TAYLOR SERIES.

For deriving finite difference relations the Taylor series expansion can be applied if the grid points are evenly-spaced.

Let the function (x) be known at evenly-spaced points and also at the mid-points of the evenly-spaced points then, the first central difference of (x) at point (i) is defined by (see Figure B.1):

$$\left.\frac{dw}{dx}\right|_i = \frac{w(x_i + h/2) - w(x_i - h/2)}{h}$$

$$= \frac{w_{i+h/2} - w_{i-h/2}}{h}$$

This can also be written as:

$$\left.\frac{dw}{dx}\right|_i = \frac{w(x_i + h) - w(x_i - h)}{2h}$$

$$= \frac{w_{i+h} - w_{i-h}}{2h}$$

$$= \frac{w_{i+1} - w_{i-1}}{2h} \qquad \text{(B.1)}$$

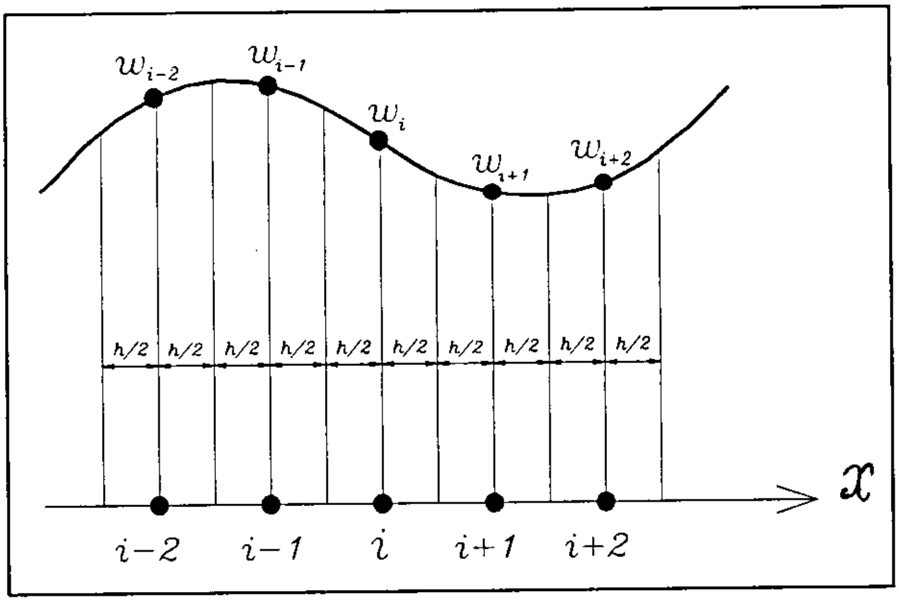

Figure (B.1) : Grid points for central difference.

The second difference at point (i) is the difference of the first difference. Hence,

$$\left.\frac{d^2w}{dx^2}\right|_i = \frac{(w_{i+\frac{1}{2}+\frac{1}{2}} - w_{i-\frac{1}{2}+\frac{1}{2}}) - (w_{i+\frac{1}{2}-\frac{1}{2}} - w_{i-\frac{1}{2}-\frac{1}{2}})}{h^2}$$

$$= \frac{w_{i+1} - 2w_i - w_{i-1}}{h^2} \qquad \text{(B.2)}$$

In the same manner, the third difference can be written as:

$$\left.\frac{d^3w}{dx^3}\right|_i = \frac{w_{i+\frac{2}{3}} - 3w_{i+\frac{1}{2}} + 3w_{i-\frac{1}{2}} - w_{i-\frac{3}{2}}}{h^3}$$

or,

$$\left.\frac{d^3w}{dx^3}\right|_i = \frac{w_{i+3} - 3w_{i+1} + 3w_{i-1} - w_{i-3}}{2h^3} \qquad \text{(B.3)}$$

Further, the forth difference is given by:

$$\left.\frac{d^3w}{dx^3}\right|_i = \frac{d^2}{dx^2}\left(\frac{d^2w}{dx^2}\right)$$

$$= \frac{\left.\frac{d^2w}{dx^2}\right|_{i+1} - 2\left.\frac{d^2w}{dx^2}\right|_i + \left.\frac{d^2w}{dx^2}\right|_{i-1}}{h^2}$$

$$= \frac{w_{i+2} - 2w_{i+1} + w_i - 2w_{i+1} + 4w_i - 2w_{i-1} + w_i - 2w_{i-1} + w_{i-2}}{h^4}$$

$$= \frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{h^4} \qquad \text{(B.4)}$$

B.3 FORMULATION OF THE PROBLEM.

Remember the non-dimensional form of the differential equation of our problem [equation (2.34)]:

$$\frac{d^4w}{d\zeta^4} + p\frac{d^2w}{d\zeta^2} + k_1 w - k_3 w^3 = 0$$

with the boundary conditions:

- For simply supported case:

$w = \frac{d^2w}{d\zeta^2} = 0$, at $\zeta = 0$ and $\zeta = 1$.

- For clamped-clamped case:

$w = \frac{dw}{d\zeta} = 0$, at $\zeta = 0$ and $\zeta = 1$.

Divide the interval (0,1) of ζ into (n) equal subintervals, each of length (h=1/n) as shown in Figure (B.2). Using the previous equations, the difference equation at any point (i) of the differential equation of the problem is given as:

$$\frac{1}{h^4}\left[w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}\right] + \frac{p}{h^2}\left[w_{i+1} - 2w_i - w_{i-1}\right] + k_1 w_i - k_3 w_i^{\,3} = 0$$

multiplying this equation by h^4, then:

$$\left[w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}\right] + ph^2\left[w_{i+1} - 2w_i - w_{i-1}\right] + k_1 h^4 w_i - k_3 h^4 w_i^{\,3} = 0 \quad \text{(B.5)}$$

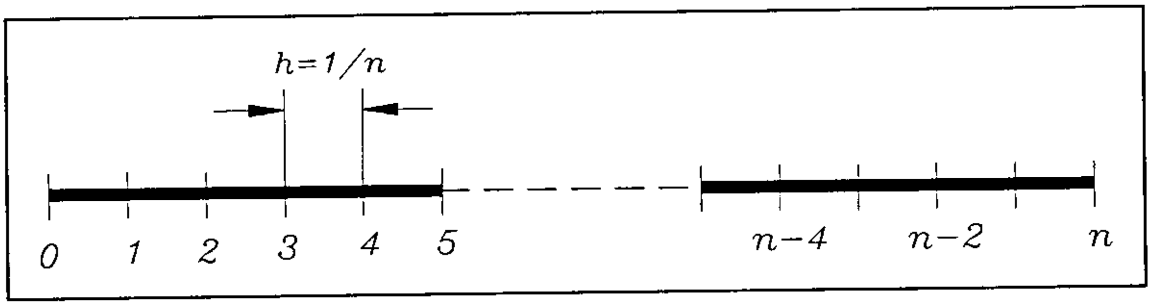

Figure (B.2): Interval divisions.

Let

$$\left.\begin{aligned} \lambda &= ph^2 = \frac{PL^2h^2}{EI} \\ \kappa_1 &= k_1 h^4 = \frac{K_1 L^4 h^4}{EI} \\ \kappa_3 &= k_3 h^4 = \frac{K_3 L^4 h^4}{EI} \end{aligned}\right\} \quad \text{(B.6)}$$

The difference equation becomes:

$$w_{i+2} + (\lambda - 4)w_{i+1} + (\kappa_1 - 2\lambda + 6)w_i + (\lambda - 4)w_{i-1} + w_{i-2} - \kappa_3 w_i^3 = 0 \quad \text{(B.7)}$$

With the coersponding boundary conditions:

- For simply supported case, and by using equation (B.2):

$$\left.\begin{aligned} w_0 &= w_n = 0 \\ w_{-1} &= w_1 \\ w_{n+1} &= w_{n-1} \end{aligned}\right\} \qquad \text{(B.8a)}$$

- For clamped-clamped case, and by using equation (B.1):

$$\left.\begin{aligned} w_0 &= w_n = 0 \\ w_{-1} &= w_1 \\ w_{n+1} &= w_{n-1} \end{aligned}\right\} \qquad \text{(B.8b)}$$

B.4 EXPRESSING THE FUNCTIONAL BY FINITE DIFFERENCE FORM

The functional of the problem, which leads - as it was seen in chapter two - to the buckling load formula [equation (3.3)]:

$$p = \frac{\int_0^1 (\frac{1}{2} w''^2 + \frac{1}{2} k_1 w^2 - \frac{1}{4} k_3 w^4) d\zeta}{\int_0^1 \frac{1}{2} w'^2 d\zeta}$$

Using the previous finite difference relations, it would be seen that:

- $\frac{1}{2}\int_0^1 w^2 d\zeta = \frac{h}{2}\sum_{i=0}^{n} w_i^2$ (B.9a)

- $\frac{1}{4}\int_0^1 w^4 d\zeta = \frac{h}{4}\sum_{i=0}^{n} w_i^4$ (B.9b)

- $$\frac{1}{2}\int_0^1 w'^2 d\zeta = \frac{h}{2}\sum_{i=0}^{n}\left(\frac{w_{i+1}-w_{i-1}}{2h}\right)^2$$

$$= \frac{1}{8h}\sum_{i=0}^{n}\left(w_{i+1}-w_{i-1}\right)^2 \qquad \text{(B.9c)}$$

- $$\frac{1}{2}\int_0^1 w''^2 d\zeta = \frac{h}{2}\sum_{i=0}^{n}\left(\frac{w_{i+1}-2w_i-w_{i-1}}{h^2}\right)^2$$

$$= \frac{1}{2h^3}\sum_{i=0}^{n}\left(w_{i+1}-2w_i-w_{i-1}\right)^2 \qquad \text{(B.9d)}$$

Subsitute equations (B.6) and (B.9) into the buckling load equation [equation (3.3)], you will end up with:

$$\lambda = \frac{4\sum_{i=0}^{n}\left(w_{i+1}-2w_i-w_{i-1}\right)^2 + 4\kappa_1\sum_{i=0}^{n} w_i^2 - 2\kappa_3\sum_{i=0}^{n} w_i^4}{\sum_{i=0}^{n}\left(w_{i+1}-w_{i-1}\right)^2}$$

$$= \frac{\sum_{i=0}^{n}[4\left(w_{i+1}-2w_i-w_{i-1}\right)^2 + 4\kappa_1 w_i^2 - 2\kappa_3 w_i^4]}{\sum_{i=0}^{n}\left(w_{i+1}-w_{i-1}\right)^2} \qquad \text{(B.10)}$$

An appropriate computer algorethem for solving non-linear system of equations can be designed based on the previous equations to end up with the solution of the problem under consideration.

APPENDIX C

NUMERICAL RESULUTS FOR THE CONVERGENCE STUDY OF THE POWER SERIES METHOD

The following tables show the numerical results for the convergence study of the solution for the clamped-clamped column on elastic foundation for different values of (k_1), (α) and starting C values.

Table (C.1) : Numerical resuluts for the convergence study showing the buckling load (p) for different polynomial degres (N) for the clamped- clamped column on elastic foundation with k_1=0, α=0 and starting C value = 0.3.

N	5	7	10	13	15	18	20	30	40
p	22.3	14.7	10	22.4	33.2	39.2	39.5	39.5	39.5

Table (C.2) : Numerical resuluts for the convergence study showing the buckling load (p) for different polynomial degres (N) for the clamped- clamped column on elastic foundation with k_1=200, α=0 and starting C value = 0.3.

N	5	6	7	8	10	15	17	19	20	25	30	40
p	24.8	50.2	50.2	23.3	31.5	42.3	57.4	57.8	62.9	62.9	62.8	63

Table (C.3) : Numerical resuluts for the convergence study showing the buckling load (p) for different polynomial degres (N) for the clamped- clamped column on elastic foundation with k_1=200, α=0.6 and starting C value = 0.3.

N	4	6	8	10	13	15	18	19	20	21	25	27
p	24.8	50	23.3	31.2	44.5	37.7	37.7	37.9	45.5	45.5	46.2	46.1

Table (C.4) : Numerical resuluts for the convergence study showing the buckling load (p) for different polynomial degres (N) for the clamped- clamped column on elastic foundation with k_1=100, α=3 and starting C value = 0.2.

N	5	7	10	13	18	19	20	22	25	30	35
p	23.3	48.8	26.2	42.1	36.2	34.8	43.7	40.1	43.5	43.4	43.5

ملخص

تحليل الالتواء غير الخطي للأعمدة المسندة على أرضية مرنة.

اعداد

هاشم سليم الخالدي

المشرف

الدكتور مازن القيسي

تتعلق هذه الرسالة بدراسة نظام الالتواء غير الخطي للأعمدة المسندة على أرضية مرنة؛ و ذلك باستخدام طريقتي متسلسلة القوى و الدالة التجربة. حيث طبقت كلتا الطريقتين لدراسة الأعمدة المثبتة الطرفين، في حين استخدمت طريقة الدالة التجريبية فقط لدراسة الأعمدة بسيطة التثبيت. و تم تطبيق مبدأ التحوير لاشتقاق المعادلة التفاضلية و إيجاد الشروط الحدية للنظام بحيث حولت هذه المعادلة وشروطها الحدية الى صيغة غير بعدية.

عند استخدام كلتا الطريقتين تمت دراسة المتغيرات و تأثيراتها: المعامل الخطي للأرضية المرنة، معامل النسبة، مقدار الانحناء و درجة متسلسلة القوى في المجال المسموح به فيزيائيا.

و اعتمادا على دراسة التقارب لمتسلسلة القوى، تم حساب قيمة حمل الالتواء بعد درجة معينة لهذه المتسلسلة، حيث لوحظ أنها مقبولة عند القيمة ٢٠.

لقد وجد أن زيادة المعامل الخطي للأرضية المرنة و تقليل معامل النسبة، يؤدي الى زيادة مقدار الحمل اللازم للي الأعمدة. كما لوحظ أيضا أن سلوك طريقتي الحل المستخدمتين كان متماثلا، و تبين أن طريقة متسلسلة القوى فعالة إذا ما برمجت.